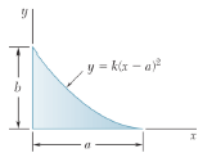
U4-Fuerzas de inercia y momentos de inercia – cap. 9

9.1-Momento de inercia de áreas

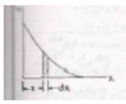
9.1-11y12va



PROBLEM 9.1

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $x=0,\ \ y=b:\ \ b=k(0-a)^2$

$k=\dfrac{b}{a^2}$ then

$$y=\frac{b}{a^2}(x-a)^2;\ dA=y\,dx$$

$$dI_y=x^2dA=x^2y\,dx$$

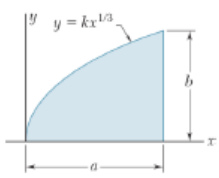
$$I_y=\int_0^a\left[\frac{b}{a^2}x^2(x-a)^2\right]dx$$

$$=\int_0^a\left[\frac{b}{a^2}\left(x^4-2ax^3+a^2x^2\right)\right]dx$$

$$=\frac{b}{a^2}\left[\frac{1}{5}x^5-\frac{1}{2}ax^4+\frac{1}{3}a^2x^3\right]_0^a$$

$$I_y=\frac{1}{30}a^3b\ ◀$$

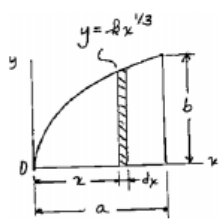
9.2-11y12va



PROBLEM 9.2

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



$$y=kx^{1/3}$$

For $x=a$:

$$b=ka^{1/3}$$

$$k=b/a^{1/3}$$

Thus:

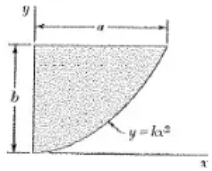
$$y=\frac{b}{a^{1/3}}x^{1/3}$$

$$dI_y=x^2dA=x^2y\,dx$$

$$dI_y=x^2\frac{b}{a^{1/3}}x^{1/3}dx=\frac{b}{a^{1/3}}x^{7/3}dx$$

$$I_y=\int dI_y=\frac{b}{a^{1/3}}\int_0^a x^{7/3}dx=\frac{b}{a^{1/3}}\left(\frac{3}{10}a^{10/3}\right)\qquad I_y=\frac{3}{10}a^3b\ ◀$$

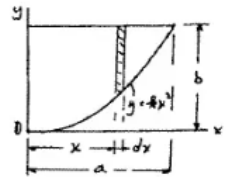
9.3-9y12va



PROBLEM 9.3

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



$$y = kx^2$$

For $x = a$:

$$b = ka^2$$

$$k = \frac{b}{a^2}$$

Thus:

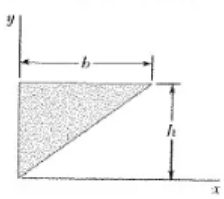
$$y = \frac{b}{a^2}x^2$$

$$dA = (b - y)dx$$

$$dI_y = x^2 dA = x^2(b - y)dx = x^2\left(b - \frac{b}{a^2}x^2\right)dx$$

$$I_y = \int dI_y = \int_0^a \left(bx^2 - \frac{b}{a^2}x^4\right)dx = \frac{1}{3}a^3b - \frac{1}{5}a^3b \qquad I_y = \frac{2}{15}a^3b \blacktriangleleft$$

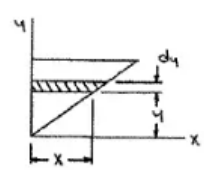
9.5-9va



PROBLEM 9.5

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



By observation

$$y = \frac{h}{b}x$$

or

$$x = \frac{b}{h}y$$

Now

$$dI_x = y^2 dA = y^2(x\,dy)$$

$$= y^2\left(\frac{b}{h}y\,dy\right)$$

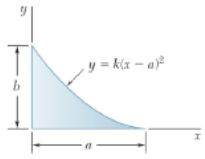
$$= \frac{b}{h}y^3 dy$$

Then

$$I_x = \int dI_x = \int_0^h \frac{b}{h}y^3 dy$$

$$= \frac{b}{h}\left[\frac{1}{4}y^4\right]_0^h \qquad \text{or} \quad I_x = \frac{1}{4}bh^3 \blacktriangleleft$$

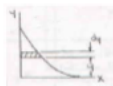
9.5-11va



PROBLEM 9.5

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



At $x=0,\ \ y=b:\ \ b=k(0-a)^2$

$k=\dfrac{b}{a^2}$ then $y=\dfrac{b}{a^2}(x-a)^2$ or $x=a\left(1\pm\dfrac{\sqrt{y}}{\sqrt{b}}\right)$, take the negative root so that $y=b$ when $x=0$.

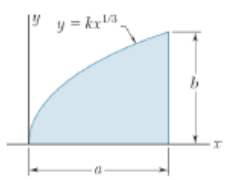
$$dI_x = y^2 dA = y^2 x\,dy$$

$$I_x = \int_0^b \left[a\left(y^2 - \frac{1}{b^{1/2}}y^{5/2}\right)\right]dy$$

$$= a\left[\frac{1}{3}y^3 - \frac{2}{7}\left(\frac{1}{b^{1/2}}\right)y^{7/2}\right]_0^b$$

$$I_x = \frac{1}{21}ab^3 \blacktriangleleft$$

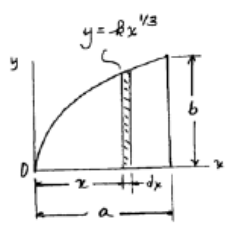
9.6-11y12va



PROBLEM 9.6

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



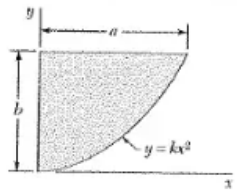
$$y = \frac{b}{a^{1/3}}x^{1/3}$$

$$dI_x = \frac{1}{3}y^3 dx = \frac{1}{3}\left(\frac{b}{a^{1/3}}x^{1/3}\right)^3 dx = \frac{1}{3}\frac{b^3}{a}x\,dx$$

$$I_x = \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a}x\,dx = \frac{1}{3}\frac{b^3}{a}\frac{a^2}{2}$$

$$I_x = \frac{1}{6}ab^3 \blacktriangleleft$$

9.7-9y12va

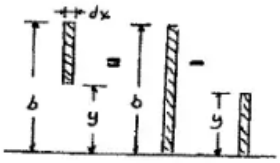


PROBLEM 9.7

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

See figure of solution of Problem 9.3.



$$y = \frac{b}{a^2}x$$

$$dI_x = \frac{1}{3}b^3dx - \frac{1}{3}y^3dx = \frac{1}{3}b^3dx - \frac{1}{3}\frac{b^3}{a^6}x^6dx$$

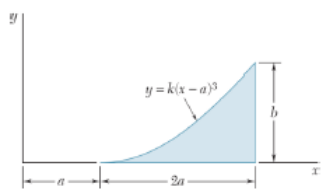
$$I_x = \int dI_x = \int_0^a \left(\frac{1}{3}b^3 - \frac{1}{3}\frac{b^3}{a^6}x^6\right)dx$$

$$= \frac{1}{3}b^3a - \frac{1}{3}\frac{b^3}{a^6}\frac{a^7}{7} = \left(\frac{1}{3} - \frac{1}{21}\right)ab^3$$

$$= \left(\frac{7}{21} - \frac{1}{21}\right)ab^3 = \frac{6}{21}ab^3$$

$$I_x = \frac{2}{7}ab^3 \blacktriangleleft$$

9.9-11y12va



PROBLEM 9.9

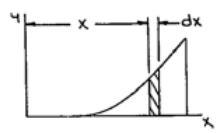
Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION

At $x = 3a$, $y = b$: $\quad b = k(3a - a)^3$

or $\quad k = \dfrac{b}{8a^3}$

Then $\quad y = \dfrac{b}{8a^3}(x - a)^3$



Now

$$dI_x = \frac{1}{3}y^3dx$$

$$= \frac{1}{3}\left[\frac{b}{8a^3}(x - a)^3\right]^3 dx$$

$$= \frac{b^3}{1536a^9}(x - a)^9 dx$$

Then

$$I_x = \int dI_x = \int_a^{3a} \frac{b^3}{1536a^9}(x - a)^9 dx = \frac{b^3}{1536a^9}\left[\frac{1}{10}(x - a)^{10}\right]_a^{3a}$$

$$= \frac{b^3}{15{,}360a^9}[(3a - a)^{10} - 0]$$

or $\quad I_x = \dfrac{1}{15}ab^3 \blacktriangleleft$

SMART BOOK

9.10-12va

$I_{y'} = \int x^2 \, dA$

$= \int_0^a x^2 \, y \, dx$

$= \int_0^a x^2 \, \frac{b}{a} \cdot x \, dx \rightarrow \frac{b}{a} \int_0^a x^3 \, dx \rightarrow \frac{b}{a} \left| \frac{x^4}{4} \right|_0^a = \frac{b}{a} \frac{a^4}{4} \rightarrow$

$\rightarrow I_{y'} = \frac{a^3 b}{4}$

$I_{x_1} = \int y^3 \frac{dx}{3} = \int_0^a \left(\frac{b}{a} x\right)^3 dx = \frac{b^3}{3a^3} \int_0^a x^3 \, dx$

$\hookrightarrow \frac{b^3}{3a^3} \left| \frac{x^4}{4} \right|_0^a$

$\hookrightarrow \frac{b^3}{3a^3} \cdot \frac{a^4}{4}$

$I_{x_1} = \frac{ab^3}{12}$

$I_{y2} = \frac{a^3 b}{5}$

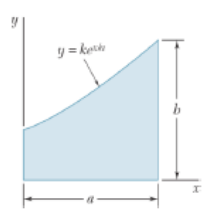
$x_x = I_{x_1} - I_{x2} = \frac{ab^3}{12} - \frac{ab^3}{21} = \frac{ab^3}{3}\left(\frac{1}{4} - \frac{1}{7}\right)$

$= \frac{ab^3}{3}\left(\frac{7-4}{28}\right) = \frac{3ab^3}{3 \cdot 28} = \frac{ab^3}{28}$

$I_y = I_{y_1} - I_{y2} = \frac{a^3 b}{4} - \frac{a^3 b}{5} = a^3 b \left(\frac{1}{4} - \frac{1}{5}\right)$

$= a^3 b \left(\frac{5-4}{20}\right) = \frac{a^3 b}{20} \; \text{o} \; \frac{ab^3}{20}$

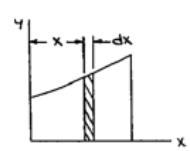
9.11-11y12va



PROBLEM 9.11

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



At $x = a$, $y = b$: $\quad b = ke^{a/a}$ or $k = \frac{b}{e}$

Then $\quad y = \frac{b}{e} e^{x/a} = be^{x/a - 1}$

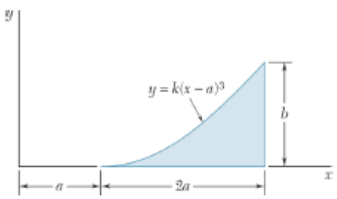
Now $\quad dI_x = \frac{1}{3} y^3 dx = \frac{1}{3}(be^{x/a-1})^3 dx$

$= \frac{1}{3} b^3 e^{3(x/a-1)} dx$

Then $\quad I_x = \int dI_x = \int_0^a \frac{1}{3} b^3 e^{3(x/a-1)} dx = \frac{b^3}{3}\left[\frac{a}{3} e^{3(x/a-1)}\right]_0^a$

$= \frac{1}{9} ab^3 (1 - e^{-3})$ $\quad$ or $\quad I_x = 0.1056ab^3$ ◀

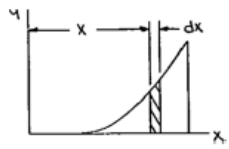
9.12-11y12va



PROBLEM 9.12

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $x = 3a, \quad y = b$: $\quad b = k(3a-a)^3$

or $\quad k = \dfrac{b}{8a^3}$

Then $\quad y = \dfrac{b}{8a^3}(x-a)^3$

Now
$$dI_y = x^2 dA = x^2(y\,dx) = x^2\left[\frac{b}{8a^3}(x-a)^3 dx\right]$$
$$= \frac{b}{8a^3}x^2(x^3 - 3x^2a + 3xa^2 - a^3)dx$$

Then
$$I_y = \int dI_y = \int_a^{3a} \frac{b}{8a^3}(x^5 - 3x^4a + 3x^3a^2 - a^3x^2)dx$$
$$= \frac{b}{8a^3}\left[\frac{1}{6}x^6 - \frac{3}{5}ax^5 + \frac{3}{4}a^2x^4 - \frac{1}{3}a^3x^3\right]_a^{3a}$$
$$= \frac{b}{8a^3}\left\{\left[\frac{1}{6}(3a)^6 - \frac{3}{5}a(3a)^5 + \frac{3}{4}a^2(3a)^4 - \frac{1}{3}a^3(3a)^3\right]\right.$$
$$\left. - \left[\frac{1}{6}(a)^6 - \frac{3}{5}a(a)^5 + \frac{3}{4}a^2(a)^4 - \frac{1}{3}a^3(a)^3\right]\right\}$$

or $\quad I_y = 3.43a^3b$ ◀

9.13-12va

13. $y(x) = b(1 - k\sqrt{x})$

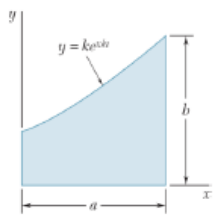
$y(a) = 0 \Rightarrow k = \dfrac{1}{\sqrt{a}}$

$$I_y = \int_0^a \int_{-y(x)}^{y(x)} x^2\,dy\,dx$$

9.13

$$= \frac{2a^3b}{21}$$

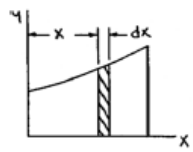
9.14-11y12va



PROBLEM 9.14

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $x = a, \quad y = b$: $\qquad b = ke^{a/a}$

or $\qquad k = \dfrac{b}{e}$

Then $\qquad y = \dfrac{b}{e}e^{x/a} = be^{x/a-1}$

Now
$$dI_y = x^2 dA = x^2(ydx)$$
$$= x^2(be^{x/a-1}dx)$$

Then
$$I_y = \int dI_y = \int_0^a bx^2 e^{x/a-1}dx$$

Now use integration by parts with

$$u = x^2 \qquad dv = e^{x/a-1}dx$$
$$du = 2xdx \qquad v = ae^{x/a-1}$$

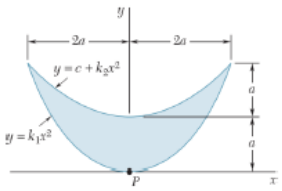
Then
$$\int_0^a x^2 e^{x/a-1}dx = \left[x^2 ae^{x/a-1}\right]_0^a - \int_0^a (ae^{x/a-1})2xdx$$
$$= a^3 - 2a\int_0^a xe^{x/a-1}dx$$

Using integration by parts with

$$u = x \qquad dv = e^{x/a-1}dx$$
$$du = dx \qquad v = ae^{x/a-1}$$

Then
$$I_y = b\left\{a^3 - 2a\left[(xae^{x/a-1})\Big|_0^a - \int_0^a (ae^{x/a-1})dx\right]\right\}$$
$$= b\left\{a^3 - 2a\left[a^2 - (a^2 e^{x/a-1})\Big|_0^a\right]\right\}$$
$$= b\left\{a^3 - 2a\left[a^2 - (a^2 - a^2 e^{-1})\right]\right\} \qquad \text{or} \quad I_y = 0.264a^3 b \blacktriangleleft$$

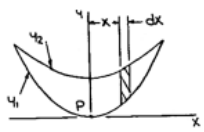
9.23-11y12va



PROBLEM 9.23

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



y_1: At $x = 2a$, $y = 2a$:

$$2a = k_1(2a)^2 \quad \text{or} \quad k_1 = \frac{1}{2a}$$

y_2: At $x = 0$, $y = a$: $\quad a = c$

At $x = 2a$, $y = 2a$:

$$2a = a + k_2(2a)^2 \quad \text{or} \quad k_2 = \frac{1}{4a}$$

Then
$$y_1 = \frac{1}{2a}x^2 \qquad y_2 = a + \frac{1}{4a}x^2 = \frac{1}{4a}(4a^2 + x^2)$$

Now
$$dA = (y_2 - y_1)dx = \left[\frac{1}{4a}(4a^2 + x^2) - \frac{1}{2a}x^2\right]dx = \frac{1}{4a}(4a^2 - x^2)dx$$

Then
$$A = \int dA = 2\int_0^{2a} \frac{1}{4a}(4a^2 - x^2)dx = \frac{1}{2a}\left[4a^2x - \frac{1}{3}x^3\right]_0^{2a} = \frac{8}{3}a^2$$

Now
$$dI_x = \left(\frac{1}{3}y_2^3 - \frac{1}{3}y_1^3\right)dx = \frac{1}{3}\left\{\left[\frac{1}{4a}(4a^2 + x^2)\right]^3 - \left[\frac{1}{2a}x^2\right]^3\right\}dx$$
$$= \frac{1}{3}\left[\frac{1}{64a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 + x^6) - \frac{1}{8a^3}x^6\right]dx$$
$$= \frac{1}{192a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 - 7x^6)dx$$

PROBLEM 9.23 (Continued)

Then
$$I_x = \int dI_x = 2\int_0^{2a} \frac{1}{192a^3}(64a^6 + 48a^4x^2 + 12a^2x^4 - 7x^6)dx$$
$$= \frac{1}{96a^3}\left[64a^6x + 16a^4x^3 + \frac{12}{5}a^2x^5 - x^7\right]_0^{2a}$$
$$= \frac{1}{96a^3}\left[64a^6(2a) + 16a^4(2a)^3 + \frac{12}{5}a^2(2a)^5 - (2a)^7\right]$$
$$= \frac{1}{96}a^4\left(128 + 128 + \frac{12}{5} \times 32 - 128\right) = \frac{32}{15}a^4$$

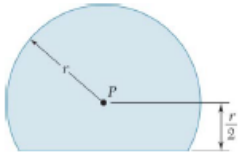
Also
$$dI_y = x^2 dA = x^2\left[\frac{1}{4a}(4a^2 - x^2)dx\right]$$

Then
$$I_y = \int dI_y = 2\int_0^{2a} \frac{1}{4a}x^2(4a^2 - x^2)dx = \frac{1}{2a}\left[\frac{4}{3}a^2x^3 - \frac{1}{5}x^5\right]_0^{2a}$$
$$= \frac{1}{2a}\left[\frac{4}{3}a^2(2a)^3 - \frac{1}{5}(2a)^5\right] = \frac{32}{2}a^4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{32}{15}a^4$$

Now
$$J_P = I_x + I_y = \frac{32}{15}a^4 + \frac{32}{15}a^4 \qquad \text{or} \quad J_P = \frac{64}{15}a^4 \blacktriangleleft$$

and
$$k_P^2 = \frac{J_P}{A} = \frac{\frac{64}{15}a^4}{\frac{8}{3}a^2} = \frac{8}{5}a^2 \qquad \text{or} \quad k_P = 1.265a \blacktriangleleft$$

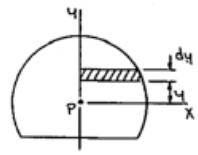
9.24-11y12va



PROBLEM 9.24

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point P.

SOLUTION



The equation of the circle is

$$x^2 + y^2 = r^2$$

So that $$x = \sqrt{r^2 - y^2}$$

Now $$dA = x\,dy = \sqrt{r^2 - y^2}\,dy$$

Then $$A = \int dA = 2\int_{-r/2}^{r} \sqrt{r^2 - y^2}\,dy$$

Let $$y = r\sin\theta; \quad dy = r\cos\theta\,d\theta$$

Then $$A = 2\int_{-\pi/6}^{\pi/2} \sqrt{r^2 - (r\sin\theta)^2}\,r\cos\,d\theta$$

$$= 2\int_{-\pi/6}^{\pi/2} r^2\cos^2\theta\,d\theta = 2r^2\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\pi/6}^{\pi/2}$$

$$= 2r^2\left[\frac{\frac{\pi}{2}}{2} - \left(\frac{-\frac{\pi}{6}}{2} + \frac{\sin -\frac{\pi}{3}}{4}\right)\right] = 2r^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right)$$

$$= 2.5274r^2$$

Now $$dI_x = y^2 dA = y^2\left(\sqrt{r^2 - y^2}\,dy\right)$$

Then $$I_x = \int dI_x = 2\int_{-r/2}^{r} y^2\sqrt{r^2 - y^2}\,dy$$

Let $$y = r\sin\theta; \quad dy = r\cos\theta\,d\theta$$

Then $$I_x = 2\int_{-\pi/6}^{\pi/2} (r\sin\theta)^2\sqrt{r^2 - (r\sin\theta)^2}\,r\cos\theta\,d\theta$$

$$= 2\int_{-\pi/6}^{\pi/2} r^2\sin^2\theta(r\cos\theta)r\cos\theta\,d\theta$$

Now $$\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin^2\theta\cos^2\theta = \frac{1}{4}\sin 2\theta$$

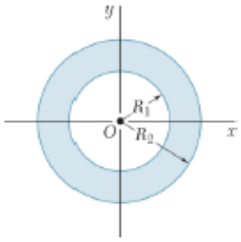
PROBLEM 9.24 (Continued)

Now $$J_P = I_x + I_y = \frac{r^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{16}\right) + \frac{2}{3}r^4\left(\frac{\pi}{4} + \frac{9\sqrt{3}}{64}\right)$$

$$= r^4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{16}\right) = 1.15545r^4 \qquad \text{or} \quad J_P = 1.155r^4 \blacktriangleleft$$

and $$k_P^2 = \frac{J_P}{A} = \frac{1.15545r^4}{2.5274r^2} \qquad \text{or} \quad k_P = 0.676r \blacktriangleleft$$

9.25-11y12va



PROBLEM 9.25

(*a*) Determine by direct integration the polar moment of inertia of the annular area shown with respect to Point *O*. (*b*) Using the result of Part *a*, determine the moment of inertia of the given area with respect to the *x* axis.

SOLUTION

(*a*)

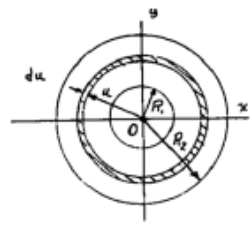
$$dA = 2\pi u\, du$$

$$dJ_O = u^2 dA = u^2 (2\pi u\, du)$$

$$= 2\pi u^3\, du$$

$$J_O = \int dJ_O = 2\pi \int_{R_1}^{R_2} u^3 du$$

$$= 2\pi \left| \frac{1}{4} u^4 \right|_{R_1}^{R_2}$$



$$J_O = \frac{\pi}{2}\left[R_2^4 - R_1^4\right] \blacktriangleleft$$

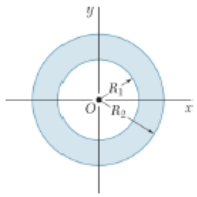
(*b*) From Eq. (9.4): (*Note* by symmetry.) $I_x = I_y$

$$J_O = I_x + I_y = 2I_x$$

$$I_x = \frac{1}{2} J_O = \frac{\pi}{4}\left[R_2^4 - R_1^4\right]$$

$$I_x = \frac{\pi}{4}\left[R_2^4 - R_1^4\right] \blacktriangleleft$$

9.26-11y12va

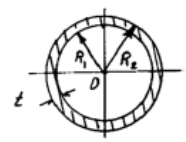


PROBLEM 9.26

(*a*) Show that the polar radius of gyration k_O of the annular area shown is approximately equal to the mean radius $R_m = (R_1 + R_2)/2$ for small values of the thickness $t = R_2 - R_1$. (*b*) Determine the percentage error introduced by using R_m in place of k_O for the following values of t/R_m: 1, $\frac{1}{2}$, and $\frac{1}{10}$.

SOLUTION

(*a*) From Problem 9.23:



$$J_O = \frac{\pi}{2}\left[R_2^4 - R_1^4\right]$$

$$k_O^2 = \frac{J_O}{A} = \frac{\frac{\pi}{2}\left[R_2^4 - R_1^4\right]}{\pi\left(R_2^2 - R_1^2\right)}$$

$$k_O^2 = \frac{1}{2}\left[R_2^2 + R_1^2\right]$$

Thickness: $t = R_2 - R_1$

Mean radius: $R_m = \frac{1}{2}(R_1 + R_2)$

Thus, $R_1 = R_m - \frac{1}{2}t$ and $R_2 = R_m + \frac{1}{2}t$

$$k_O^2 = \frac{1}{2}\left[\left(R_m + \frac{1}{2}t\right)^2 + \left(R_m - \frac{1}{2}t\right)^2\right] = R_m^2 + \frac{1}{4}t^2$$

For t small compared to R_m: $k_O^2 \approx R_m^2$ $\qquad k_O \approx R_m$ ◀

(*b*) Percentage error $= \dfrac{\text{(exact value)} - \text{(approximation value)}}{\text{(exact value)}}100$

$$\text{P.E.} = -\frac{\sqrt{R_m^2 + \frac{1}{4}t^2} - R_m}{\sqrt{R_m^2 + \frac{1}{4}t^2}}(100) = -\frac{\sqrt{1 + \frac{1}{4}\left(\frac{t}{R_m}\right)^2} - 1}{\sqrt{1 + \frac{1}{4}\left(\frac{t}{R_m}\right)^2}}100$$

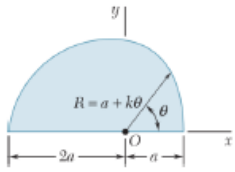
PROBLEM 9.26 (Continued)

For $\dfrac{t}{R_m} = 1$: $\qquad \text{P.E.} = \dfrac{\sqrt{1 + \frac{1}{4}} - 1}{\sqrt{1 + \frac{1}{4}}}(100) = -10.56\%$ ◀

For $\dfrac{t}{R_m} = \dfrac{1}{2}$: $\qquad \text{P.E.} = -\dfrac{\sqrt{1 + \frac{1}{4}\left(\frac{1}{2}\right)^2} - 1}{\sqrt{1 + \frac{1}{4}\left(\frac{1}{2}\right)^2}}(100) = -2.99\%$ ◀

For $\dfrac{t}{R_m} = \dfrac{1}{10}$: $\qquad \text{P.E.} = \dfrac{\sqrt{1 + \frac{1}{4}\left(\frac{1}{10}\right)^2} - 1}{\sqrt{1 + \frac{1}{4}\left(\frac{1}{10}\right)^2}}(100) = -0.1250\%$ ◀

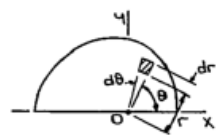
9.27-9,11y12va



PROBLEM 9.27

Determine the polar moment of inertia and the polar radius of gyration of the shaded area shown with respect to Point O.

SOLUTION



At $\theta = \pi, \quad R = 2a$: $\qquad 2a = a + k(\pi)$

or $\qquad k = \dfrac{a}{\pi}$

Then $\qquad R = a + \dfrac{a}{\pi}\theta = a\left(1 + \dfrac{\theta}{\pi}\right)$

Now
$$dA = (dr)(r\,d\theta) = r\,dr\,d\theta$$

Then
$$A = \int dA = \int_0^{\pi}\int_0^{a(1+\theta/\pi)} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}r^2\right]_0^{a(1+\theta/\pi)} d\theta$$

$$A = \int_0^{\pi}\frac{1}{2}a^2\left(1+\frac{\theta}{\pi}\right)^2 d\theta = \frac{1}{2}a^2\left[\frac{\pi}{3}\left(1+\frac{\theta}{\pi}\right)^3\right]_0^{\pi}$$

$$= \frac{1}{6}\pi a^2\left[\left(1+\frac{\pi}{\pi}\right)^3 - (1)^3\right] = \frac{7}{6}\pi a^2$$

Now $\qquad dJ_O = r^2 dA = r^2(r\,dr\,d\theta)$

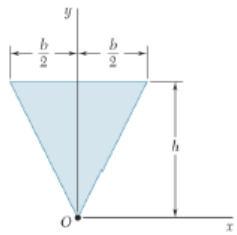
Then
$$J_O = \int dJ_O = \int_0^{\pi}\int_0^{a(1+\theta/\pi)} r^3\,dr\,d\theta$$

$$= \int_0^{\pi}\left[\frac{1}{4}r^4\right]_0^{a(1+\theta/\pi)} d\theta = \int_0^{\pi}\frac{1}{4}a^4\left(1+\frac{\theta}{\pi}\right)^4 d\theta$$

$$= \frac{1}{4}a^4\left[\frac{\pi}{5}\left(1+\frac{\theta}{\pi}\right)^5\right]_0^{\pi} = \frac{1}{20}\pi a^4\left[\left(1+\frac{\pi}{\pi}\right)^5 - (1)^5\right] \quad \text{or} \quad J_O = \frac{31}{20}\pi a^4 \blacktriangleleft$$

and
$$k_O^2 = \frac{J_O}{A} = \frac{\frac{31}{20}\pi a^4}{\frac{7}{6}\pi a^2} = \frac{93}{70}a^2 \qquad \text{or} \quad k_O = 1.153a \blacktriangleleft$$

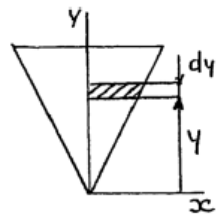
9.28-9,11y12va



PROBLEM 9.28

Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to Point O.

SOLUTION



By observation: $y = \dfrac{h}{\frac{b}{2}} x$

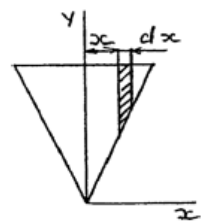
or $x = \dfrac{b}{2h} y$

Now $dA = x dy = \left(\dfrac{b}{2h} y\right) dy$

and $dI_x = y^2 dA = \dfrac{b}{2h} y^3 dy$

Then $I_x = \int dI_x = 2\int_0^h \dfrac{b}{2h} y^3 dy$

$$= \frac{b}{h} \frac{y^4}{4}\bigg|_0^h = \frac{1}{4} bh^3$$



From above: $y = \dfrac{2h}{b} x$

Now $dA = (h - y) dx = \left(h - \dfrac{2h}{b} x\right) dx$

$$= \frac{h}{b}(b - 2x) dx$$

and $dI_y = x^2 dA = x^2 \dfrac{h}{b}(b - 2x) dx$

PROBLEM 9.28 (Continued)

Then $I_y = \int dI_y = 2\int_0^{b/2} \dfrac{h}{b} x^2 (b - 2x) dx$

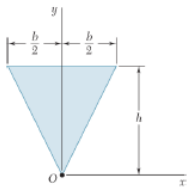
$$= 2\frac{h}{b}\left[\frac{1}{3} bx^3 - \frac{1}{2} x^4\right]_0^{b/2}$$

$$= 2\frac{h}{b}\left[\frac{b}{3}\left(\frac{b}{2}\right)^3 - \frac{1}{2}\left(\frac{b}{2}\right)^4\right] = \frac{1}{48} b^3 h$$

Now $J_O = I_x + I_y = \dfrac{1}{4} bh^3 + \dfrac{1}{48} b^3 h$ or $J_O = \dfrac{bh}{48}(12h^2 + b^2)$ ◀

and $k_O^2 = \dfrac{J_O}{A} = \dfrac{\frac{bh}{48}(12h^2 + b^2)}{\frac{1}{2} bh} = \dfrac{1}{24}(12h^2 + b^2)$ or $k_O = \dfrac{\sqrt{12h^2 + b^2}}{24}$ ◀

9.29-11y12va



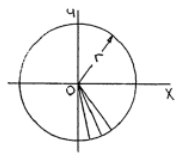
PROBLEM 9.29*

Using the polar moment of inertia of the isosceles triangle of Problem 9.28, show that the centroidal polar moment of inertia of a circular area of radius r is $\pi r^4/2$. (*Hint:* As a circular area is divided into an increasing number of equal circular sectors, what is the approximate shape of each circular sector?)

PROBLEM 9.28 Determine the polar moment of inertia and the polar radius of gyration of the isosceles triangle shown with respect to Point O.

SOLUTION

First the circular area is divided into an increasing number of identical circular sectors. The sectors can be approximated by isosceles triangles. For a large number of sectors the approximate dimensions of one of the isosceles triangles are as shown.



For an isosceles triangle (see Problem 9.28):

$$J_O = \frac{bh}{48}(12h^2 + b^2)$$

Then with $\quad b = r\Delta\theta \quad$ and $\quad h = r$

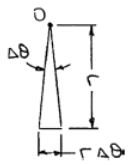
$$(\Delta J_O)_{\text{sector}} \simeq \frac{1}{48}(r\Delta\theta)(r)[12r^2 + (r\Delta\theta)^2]$$

$$= \frac{1}{48}r^4\Delta\theta\left[(12 + \Delta\theta^2)\right]$$

Now

$$\frac{dJ_{O\,\text{sector}}}{d\theta} = \lim_{\Delta\theta\to 0}\left(\frac{\Delta J_{O\,\text{sector}}}{\Delta\theta}\right) = \lim_{\Delta\theta\to 0}\left\{\frac{1}{48}r^4[12 + (\Delta\theta)^2]\right\}$$

$$= \frac{1}{4}r^4$$



Then

$$(J_O)_{\text{circle}} = \int dJ_{O\,\text{sector}} = \int_0^{2\pi}\frac{1}{4}r^4 d\theta = \frac{1}{4}r^4[\theta]_0^{2\pi}$$

or $\quad (J_O)_{\text{circle}} = \frac{\pi}{2}r^4$ ◀

Teorema de los ejes paralelos y áreas compuestas

9.31-12va

$$I_{x-x} = \int y^2 dA$$

$$= 2\left[\int_0^5 y^2 dA + \int_5^{45} y^2 dA\right]$$

$$0 \le x \le 5$$

$$5 \le x \le 45$$

re

$$= 2\left[\int_0^5 y^2(90\ dy) + \int_5^{45} y^2(10\ dy)\right]$$

$$= 2\left[90\left[\frac{y^{2+1}}{2+1}\right]_0^5 + 10\left[\frac{y^{2+1}}{2+1}\right]_5^{45}\right]$$

$$= \frac{1842500}{3}$$

$$= 614166.67\,\text{mm}^4$$

Therefore, the moment of inertia with respect to the x-axis is:

$$\boxed{I_{x-x} = 614166.67\,\text{mm}^4}$$

9.31

9.33-12va

$$
\begin{aligned}
I_{y-y} &= \int x^2 dA \\
&= 2\left[\int_0^{35} x^2 dA + \int_{35}^{45} x^2 dA\right] \\
&0 \le x \le 35 \\
&35 \le x \le 45 \\
&= 2\left[\int_0^{35} x^2 (10\ dx) + \int_{35}^{45} x^2 (50\ dx)\right] \\
&= 2\left[10\left[\frac{x^{2+1}}{2+1}\right]_0^{35} + 50\left[\frac{x^{2+1}}{2+1}\right]_{35}^{45}\right] \\
&= \frac{5682500}{3} \\
&= 1894166.67\,\mathrm{mm}^4
\end{aligned}
$$

Therefore, the moment of inertia with respect to the *y*-axis is:

$$\boxed{I_{y-y} = 1894166.67\,\mathrm{mm}^4}$$

9.33

$$
\begin{aligned}
A &= 2\,(10 \times 35 + 50 \times 10) \\
&= 1700\,\mathrm{mm}^2 \\
I_{x-x} &= 614166.66\,\mathrm{mm}^4 \\
\Rightarrow Ak_{x-x}^2 &= I_{x-x} \\
\Rightarrow (1700)k_{x-x}^2 &= 614166.67 \\
\Rightarrow k_{x-x} &= 19.01\mathrm{mm} \\
I_{y-y} &= \frac{5682500}{3}\,\mathrm{mm}^4 \\
\Rightarrow Ak_{y-y}^2 &= I_{y-y} \\
\Rightarrow (1700)k_{y-y}^2 &= \frac{5682500}{3} \\
\Rightarrow k_{y-y} &= 33.38\mathrm{mm}
\end{aligned}
$$

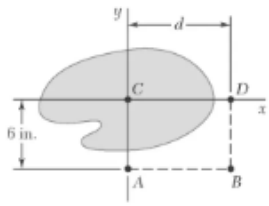
Therefore, the radius of gyration of the shaded area with respect to the *x* and *y*-axis are:

$$\boxed{\begin{aligned} k_{x-x} &= 19.01\mathrm{mm} \\ k_{y-y} &= 33.38\mathrm{mm} \end{aligned}}$$

9.31

9.33

9.39-9va (parecida a la 12va, pero con pulgadas)



PROBLEM 9.39

The shaded area is equal to 50 in.^{2}. Determine its centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$, knowing that $\bar{I}_y = 2\bar{I}_x$ and that the polar moment of inertia of the area about Point A is $J_A = 2250$ in.^{4}.

SOLUTION

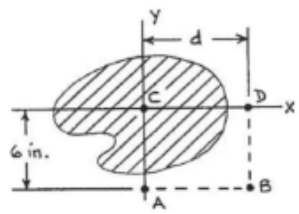
Given: $A = 50$ in.^{2} $\quad \bar{I}_y = 2\bar{I}_x, \quad J_A = 2250$ in.^{4}

$$J_A = \bar{J}_C + A(6 \text{ in.})^2$$

$$2250 \text{ in.}^4 = \bar{J}_C + (50 \text{ in.}^2)(6 \text{ in.})^2$$

$$\bar{J}_C = 450 \text{ in.}^4$$

$$\bar{J}_C = \bar{I}_x + \bar{I}_y \quad \text{with} \quad \bar{I}_y = 2\bar{I}_x$$

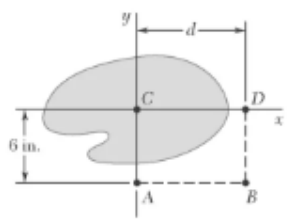


$$450 \text{ in.}^4 = \bar{I}_x + 2\bar{I}_x$$

$\bar{I}_x = 150.0$ in.^{4} ◀

$\bar{I}_y = 2\bar{I}_x = 300$ in.^{4} ◀

9.40-9va (parecida a la 12va, pero con pulgadas)



PROBLEM 9.40

The polar moments of inertia of the shaded area with respect to Points A, B, and D are, respectively, $J_A = 2880$ in.^{4}, $J_B = 6720$ in.^{4}, and $J_D = 4560$ in.^{4}. Determine the shaded area, its centroidal moment of inertia $\bar{J}_C$, and the distance d from C to D.

SOLUTION

See figure at solution of Problem 9.39.

Given: $J_A = 2880$ in.^{4}, $\quad J_B = 6720$ in.^{4}, $\quad J_D = 4560$ in.^{4}

$$J_B = \bar{J}_C + A(CB)^2; \quad 6720 \text{ in.}^4 = \bar{J}_C + A(6^2 + d^2) \qquad (1)$$

$$J_D = \bar{J}_C + A(CD)^2; \quad 4560 \text{ in.}^4 = \bar{J}_C + Ad^2 \qquad (2)$$

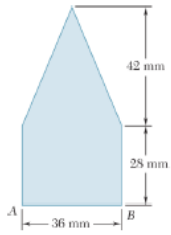
Eq. (1) subtracted by Eq. (2): $J_B - J_D = 2160 \text{ in.}^4 = A(6)^2$ $\qquad A = 60.0$ in.^{2} ◀

$$J_A = \bar{J}_C + A(AC)^2; \quad 2880 \text{ in.}^4 = \bar{J}_C + (60 \text{ in.}^2)(6 \text{ in.})^2$$

$\bar{J}_C = 720$ in.^{4} ◀

Eq. (2): $\quad 4560 \text{ in.}^4 = 720 \text{ in.}^4 + (60 \text{ in.}^2)d^2$ $\qquad d = 8.00$ in. ◀

9.42-11y12va



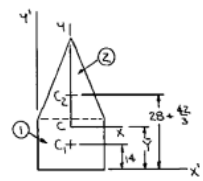
PROBLEM 9.42

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION

First locate C of the area.

Symmetry implies $\bar{X} = 18$ mm.



Dimensions in mm

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$36 \times 28 = 1008$	14	14,112
2	$\frac{1}{2} \times 36 \times 42 = 756$	42	31,752
Σ	1764		45,864

Then $\bar{Y}\Sigma A = \Sigma \bar{y}A$: $\bar{Y}(1764 \text{ mm}^2) = 45{,}864 \text{ mm}^3$

or $\bar{Y} = 26.0$ mm

Now $\bar{I}_x = (I_x)_1 + (I_x)_2$

where $(I_x)_1 = \frac{1}{12}(36 \text{ mm})(28 \text{ mm})^3 + (1008 \text{ mm}^2)[(26-14)\text{ mm}]^2$

$= (65{,}856 + 145{,}152)\text{ mm}^4 = 211.008 \times 10^3 \text{ mm}^4$

$(I_x)_2 = \frac{1}{36}(36 \text{ mm})(42 \text{ mm})^3 + (756 \text{ mm}^2)[(42-26)\text{ mm}]^2$

$= (74{,}088 + 193{,}536)\text{ mm}^4 = 267.624 \times 10^3 \text{ mm}^4$

Then $\bar{I}_x = (211.008 + 267.624) \times 10^3 \text{ mm}^4$

or $\bar{I}_x = 479 \times 10^3 \text{ mm}^4$ ◀

PROBLEM 9.42 (Continued)

Also $\bar{I}_y = (I_y)_1 + (I_y)_2$

where $(I_y)_1 = \frac{1}{12}(28 \text{ mm})(36 \text{ mm})^3 = 108.864 \times 10^3 \text{ mm}^4$

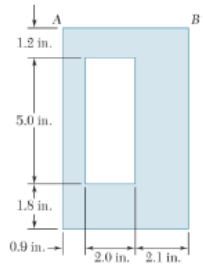
$(I_y)_2 = 2\left[\frac{1}{36}(42 \text{ mm})(18 \text{ mm})^3 + \left(\frac{1}{2} \times 18 \text{ mm} \times 42 \text{ mm}\right)(6 \text{ mm})^2\right]$

$= 2(6804 + 13{,}608)\text{ mm}^4 = 40.824 \times 10^3 \text{ mm}^4$

[$(I_y)_2$ is obtained by dividing A_2 into ◭◮]

Then $\bar{I}_y = (108.864 + 40.824 \times 10^3 \text{ mm}^4$ or $\bar{I}_y = 149.7 \times 10^3 \text{ mm}^4$ ◀

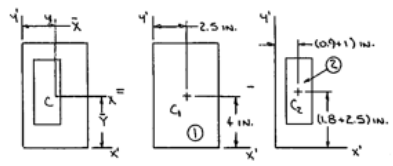
9.43-9,11y12va



PROBLEM 9.43

Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION



First locate centroid C of the area.

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$5\times 8 = 40$	2.5	4	100	160
2	$-2\times 5 = -10$	1.9	4.3	−19	−43
Σ	30			81	117

Then $\bar{X}\Sigma A = \Sigma\bar{x}A$: $\bar{X}(30\text{ in}^2) = 81\text{ in}^3$

or $\bar{X} = 2.70\text{ in.}$

and $\bar{Y}\Sigma A = \Sigma\bar{y}A$: $\bar{Y}(30\text{ in}^2) = 117\text{ in}^3$

or $\bar{Y} = 3.90\text{ in.}$

Now $\bar{I}_x = (I_x)_1 - (I_x)_2$

where
$$(I_x)_1 = \frac{1}{12}(5\text{ in.})(8\text{ in.})^3 + (40\text{ in}^2)[(4-3.9)\text{ in.}]^2$$
$$= (213.33 + 0.4)\text{ in}^4 = 213.73\text{ in}^4$$
$$(I_x)_2 = \frac{1}{12}(2\text{ in.})(5\text{ in.})^3 + (10\text{ in}^2)[(4.3-3.9)\text{ in.}]^2$$
$$= (20.83 + 1.60) = 22.43\text{ in}^4$$

PROBLEM 9.43 (Continued)

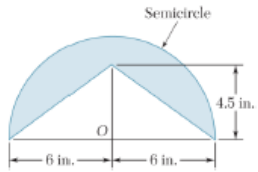
Then $\bar{I}_x = (213.73 - 22.43)\text{ in}^4$ or $\bar{I}_x = 191.3\text{ in}^4$ ◀

Also $\bar{I}_y = (I_y)_1 - (I_y)_2$

where
$$(I_y)_1 = \frac{1}{12}(8\text{ in.})(5\text{ in.})^3 + (40\text{ in}^2)[(2.7-2.5)\text{ in.}]^2$$
$$= (83.333 + 1.6)\text{ in}^4 = 84.933\text{ in}^4$$
$$(I_y)_2 = \frac{1}{12}(5\text{ in.})(2\text{ in.})^3 + (10\text{ in}^2)[(2.7-1.9)\text{ in.}]^2$$
$$= (3.333 + 6.4)\text{ in}^4 = 9.733\text{ in}^4$$

Then $\bar{I}_y = (84.933 - 9.733)\text{ in}^4$ or $\bar{I}_y = 75.2\text{ in}^4$ ◀

9.45-11y12va

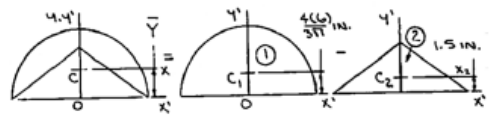


PROBLEM 9.45

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

First locate centroid *C* of the figure.



	A, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
1	$\frac{\pi}{2}(6)^2 = 56.5487$	$\frac{8}{\pi} = 2.5465$	144
2	$-\frac{1}{2}(12)(4.5) = -27$	1.5	−40.5
Σ	29.5487		103.5

Then $\bar{Y}\Sigma A = \Sigma \bar{y}A$: $\bar{Y}(29.5487 \text{ in}^2) = 103.5 \text{ in}^3$

or $\bar{Y} = 3.5027 \text{ in.}$

(*a*) $J_O = (J_O)_1 - (J_O)_2$

where $(J_O)_1 = \frac{\pi}{4}(6 \text{ in.})^4 = 107.876 \text{ in}^4$

$(J_O)_2 = (I_{x'})_2 + (I_{y'})_2$

Now $(I_{x'})_2 = \frac{1}{12}(12 \text{ in.})(4.5 \text{ in.})^3 = 91.125 \text{ in}^4$

and $(I_{y'})_2 = 2\left[\frac{1}{12}(4.5 \text{ in.})(6 \text{ in.})^3\right] = 162.0 \text{ in}^4$

[*Note:* $(I_{y'})_2$ is obtained using ◿◺]

PROBLEM 9.45 (Continued)

Then $(J_O)_2 = (91.125 + 162.0) \text{ in}^4$

$= 253.125 \text{ in}^4$

Finally $J_O = (1017.876 - 253.125) \text{ in}^4 = 764.751 \text{ in}^4$

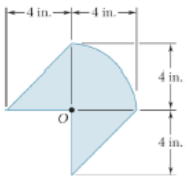
or $J_O = 765 \text{ in}^4$ ◀

(*b*) $J_O = \bar{J}_C + A\bar{Y}^2$

or $J_C = 764.751 \text{ in.}^4 - (29.5487 \text{ in}^2)(3.5027 \text{ in.})^2$

or $\bar{J}_C = 402 \text{ in}^4$ ◀

9.46-11y12va



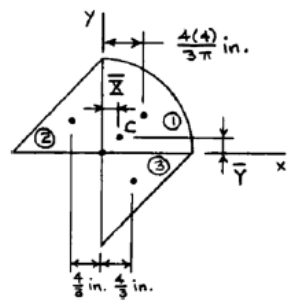
PROBLEM 9.46

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

Determination of centroid *C* of entire section:

Section	Area, in^2	$\bar{x}$, in.	$\bar{x}A$, in^3
1	$\frac{\pi}{4}(4)^2 = 4\pi$	$\frac{16}{3\pi}$	21.333
2	$\frac{1}{2}(4)(4) = 8$	$-\frac{4}{3}$	−10.6667
3	$\frac{1}{2}(4)(4) = 8$	$\frac{4}{3}$	10.6667
Σ	28.566		21.333



$$\bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(28.566\ in^2) = 21.333\ in^3$$

$$\bar{X} = 0.74680\ in.; \quad \text{by symmetry,} \quad \bar{Y} = \bar{X} = 0.74680\ in.$$

(*a*) For section ①: $I_x = \frac{1}{4}\left(\frac{1}{4}\pi r^4\right) = \frac{1}{16}\pi(4)^4 = 50.265\ in^4$

For sections ②, ③: $I_x = \frac{1}{12}bh^3 = \frac{1}{12}(4)(4)^3 = 21.333\ in^4$

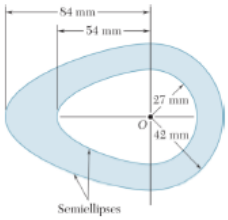
For total area, $I_x = 50.265 + 2(21.333) = 92.931\ in^4$

By symmetry, $I_y = I_x; \quad J_O = I_x + I_y = 185.862\ in^4$ $\qquad J_O = 185.9\ in^4$ ◀

(*b*) Parallel axis theorem: $J_C = J_O - A(OC)^2 = J_O - A(\bar{X}^2 + \bar{Y}^2)$

$J_C = 185.862 - (28.566)(0.74680^2 + 0.74680^2)$ $\qquad J_C = 154.0\ in^4$ ◀

9.48-11y12va



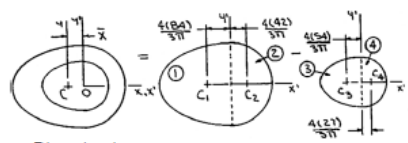
PROBLEM 9.48

Determine the polar moment of inertia of the area shown with respect to (*a*) Point *O*, (*b*) the centroid of the area.

SOLUTION

First locate centroid *C* of the figure.

Note that symmetry implies $\bar{Y} = 0$.



Dimensions in mm

	A, mm^2	$\bar{X}$, mm	$\bar{X}A$, mm^3
1	$\frac{\pi}{2}(84)(42) = 5541.77$	$-\frac{112}{\pi} = -35.6507$	−197,568
2	$\frac{\pi}{2}(42)^2 = 2770.88$	$\frac{56}{\pi} = 17.8254$	49,392
3	$-\frac{\pi}{2}(54)(27) = -2290.22$	$-\frac{72}{\pi} = -22.9183$	52,488
4	$-\frac{\pi}{2}(27)^2 = -1145.11$	$\frac{36}{\pi} = 11.4592$	−13,122
Σ	4877.32		−108,810

Then $\bar{X}\Sigma A = \Sigma\bar{X}A$: $\bar{X}(4877.32 \text{ mm}^2) = -108{,}810 \text{ mm}^3$

or $\bar{X} = -22.3094$

(*a*) $J_O = (J_O)_1 + (J_O)_2 - (J_O)_3 - (J_O)_4$

where $(J_O)_1 = \frac{\pi}{8}(84 \text{ mm})(42 \text{ mm})[(84 \text{ mm})^2 + (42 \text{ mm})^2]$

$= 12.21960 \times 10^6 \text{ mm}^4$

PROBLEM 9.48 (Continued)

$(J_O)_2 = \frac{\pi}{4}(42 \text{ mm})^4$

$= 2.44392 \times 10^6 \text{ mm}^4$

$(J_O)_3 = \frac{\pi}{8}(54 \text{ mm})(27 \text{ mm})[(54 \text{ mm})^2 + (27 \text{ mm})^2]$

$= 2.08696 \times 10^6 \text{ mm}^4$

$(J_O)_4 = \frac{\pi}{4}(27 \text{ mm})^4$

$= 0.41739 \times 10^6 \text{ mm}^4$

Then $J_O = (12.21960 + 2.44392 - 2.08696 - 0.41739) \times 10^6 \text{ mm}^4$

$= 12.15917 \times 10^6 \text{ mm}^4$

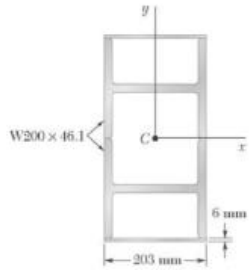
or $J_O = 12.16 \times 10^6 \text{ mm}^4$ ◀

(*b*) $J_O = \bar{J}_C + A\bar{X}^2$

or $\bar{J}_C = 12.15917 \times 10^6 \text{ mm}^4 - (4877.32 \text{ mm}^2)(-22.3094 \text{ mm})^2$

or $\bar{J}_C = 9.73 \times 10^6 \text{ mm}^4$ ◀

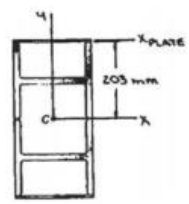
9.49-12va



PROBLEM 9.51

To form a reinforced box section, two rolled W sections and two plates are welded together. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal axes shown.

SOLUTION



W section:

$$A = 5880 \text{ mm}^2$$
$$\bar{I}_x = 15.4 \times 10^6 \text{ mm}^4$$
$$\bar{I}_y = 45.8 \times 10^6 \text{ mm}^4$$

Note:

$$A_{\text{total}} = 2A_W + 2A_{\text{plate}}$$
$$= 2(5880 \text{ mm}^2) + 2(203 \text{ mm})(6 \text{ mm})$$
$$= 14{,}196 \text{ mm}^2$$

Now

$$\bar{I}_x = 2(I_x)_W + 2(I_x)_{\text{plate}}$$

where

$$(I_x)_W = \bar{I}_x + Ad^2 = 15.4 \times 10^6 \text{ mm}^4 + (5880 \text{ mm}^2)\left(\frac{203}{2} \text{ mm}\right)^2$$
$$= 75.9772 \times 10^6 \text{ mm}^4$$
$$(I_x)_{\text{plate}} = \bar{I}_{x_{\text{plate}}} + Ad^2$$
$$= \frac{1}{12}(203 \text{ mm})(6 \text{ mm})^3 + (1218 \text{ mm}^2)[(203+3) \text{ mm}]^2$$
$$= 51.6907 \times 10^6 \text{ mm}^4$$

Then

$$\bar{I}_x = [2(75.9772) + 2(51.6907)] \times 10^6 \text{ mm}^4$$
$$= 255.336 \times 10^6 \text{ mm}^4$$

or $\bar{I}_x = 255 \times 10^6 \text{ mm}^4$ ◀

and

$$\bar{k}_x^2 = \frac{\bar{I}_x}{A_{\text{total}}} = \frac{255.336 \times 10^6 \text{ mm}^4}{14{,}196 \text{ mm}^2}$$

or $\bar{k}_x = 134.1 \text{ mm}$ ◀

PROBLEM 9.51 (Continued)

also

$$\bar{I}_y = 2(I_y)_W + 2(I_y)_{\text{plate}}$$

where

$$(I_y)_W = I_y$$
$$(I_y)_{\text{plate}} = \frac{1}{12}(6 \text{ mm})(203 \text{ mm})^3 = 4.1827 \times 10^6 \text{ mm}^4$$

Then

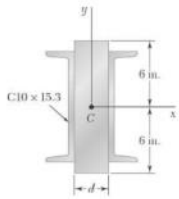
$$\bar{I}_y = [2(45.8) + 2(4.1827)] \times 10^6 \text{ mm}^4$$
$$= 99.9654 \times 10^6 \text{ mm}^4$$

or $\bar{I}_y = 100.0 \times 10^6 \text{ mm}^4$ ◀

and

$$\bar{k}_y^2 = \frac{\bar{I}_y}{A_{\text{total}}} = \frac{99.9654 \times 10^6 \text{ mm}^4}{14{,}196 \text{ mm}^2}$$

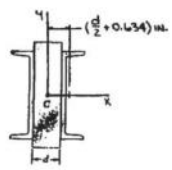
or $\bar{k}_y = 83.9 \text{ mm}$ ◀



PROBLEM 9.52

Two channels are welded to a $d \times 12$-in. steel plate as shown. Determine the width d for which the ratio $\bar{I}_x/\bar{I}_y$ of the centroidal moments of inertia of the section is 16.

SOLUTION



Channel: $A = 4.48 \text{ in.}^2$

$\bar{I}_x = 67.3 \text{ in.}^4$

$\bar{I}_y = 2.27 \text{ in.}^4$

Now $\bar{I}_x = 2(\bar{I}_x)_C + (\bar{I}_x)_{\text{plate}}$

$= 2(67.3 \text{ in.}^4) + \frac{1}{12}(d \text{ in.})(12 \text{ in.})^3$

$= (134.6 + 144d) \text{ in.}^4$

and $\bar{I}_y = 2(\bar{I}_y)_C + (\bar{I}_y)_{\text{plate}}$

where $(I_y)_C = \bar{I}_y + Ad^2$

$(I_y)_C = 2.27 \text{ in.}^4 + (4.48 \text{ in.}^2)\left[\left(\frac{d}{2} + 0.634\right) \text{in.}\right]^2$

$= (1.1200d^2 + 2.84032d + 4.07076) \text{ in.}^4$

$(\bar{I}_y)_{\text{plate}} = \frac{1}{12}(12 \text{ in.})(d \text{ in.})^3 = d^3 \text{ in.}^4$

Then $\bar{I}_y = [2(1.1200d^2 + 2.84032d + 4.07076) + d^3] \text{ in.}^4$

$= (d^3 + 2.240d^2 + 5.68064d + 8.14152) \text{ in.}^4$

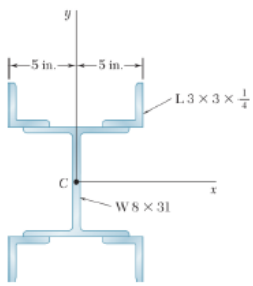
Now $\frac{\bar{I}_x}{\bar{I}_y} = 16: \quad (134.6 + 144d) = 16(d^3 + 2.240d^2 + 5.68064d + 8.14152)$

or $d^3 + 2.240d^2 - 3.31936d - 0.27098 = 0$

Solving yields $d = 1.07669$ in. (the other roots are negative).

$d = 1.077 \text{ in.}$ ◀

9.50-12va



PROBLEM 9.51

Two L6 × 4 × $\frac{1}{2}$-in. angles are welded together to form the section shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

SOLUTION

W section: $A = 9.12 \text{ in}^2 \quad \bar{I}_x = 110 \text{ in}^4 \quad \bar{I}_y = 37.1 \text{ in}^4$

Angle: $A = 1.44 \text{ in}^2 \quad \bar{I}_x = 1.23 \text{ in}^4 \quad \bar{I}_y = 1.23 \text{ in}^4$

$A_{\text{total}} = A_W + 4A_A$

$= 9.12 + 4(1.44) = 14.880 \text{ in}^2$

Now $\bar{I}_x = (\bar{I}_x)_W + 4(I_x)_A$

where $(I_x)_A = \bar{I}_{x_A} + Ad^2$

$= 1.23 \text{ in}^4 + (1.44 \text{ in}^2)(4.00 \text{ in.} + 0.836 \text{ in.})^2 = 34.907 \text{ in}^4$

Then $\bar{I}_x = (110 + 4 \times 34.907) \text{ in}^4 = 249.63 \text{ in}^4$ $\quad \bar{I}_x = 250 \text{ in}^4$ ◀

and $\bar{k}_x^2 = \frac{\bar{I}_x}{A_{\text{total}}} = \frac{249.63 \text{ in}^4}{14.880 \text{ in}^2}$ $\quad \bar{k}_x = 4.10 \text{ in.}$ ◀

Also $\bar{I}_y = (\bar{I}_y)_W + 4(\bar{I}_y)_A$

where $(\bar{I}_y)_A = \bar{I}_{y_A} + Ad^2$

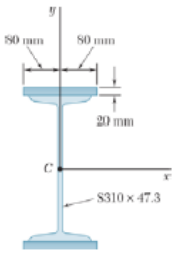
$= (1.23 + 1.44(5 - 0.836)^2) \text{ in}^4 = 26.198 \text{ in}^4$

$\bar{I}_y = (37.1 + 4 \times 26.198) \text{ in}^4 = 141.892 \text{ in}^4$ $\quad \bar{I}_y = 141.9 \text{ in}^4$ ◀

$\bar{k}_y^2 = \frac{\bar{I}_y}{A_{\text{total}}} = \frac{141.892 \text{ in}^4}{14.880 \text{ in}^2}$ $\quad \bar{k}_y = 3.09 \text{ in.}$ ◀

9.51-11y12va

9.52-11y12va



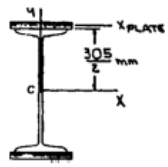
PROBLEM 9.52

Two 20-mm steel plates are welded to a rolled S section as shown. Determine the moments of inertia and the radii of gyration of the combined section with respect to the centroidal x and y axes.

SOLUTION

S section:

$$A = 6010 \text{ mm}^2$$
$$\bar{I}_x = 90.3 \times 10^6 \text{ mm}^4$$
$$\bar{I}_y = 3.88 \times 10^6 \text{ mm}^4$$



Note:

$$A_{\text{total}} = A_S + 2A_{\text{plate}}$$
$$= 6010 \text{ mm}^2 + 2(160 \text{ mm})(20 \text{ mm})$$
$$= 12{,}410 \text{ mm}^2$$

Now

$$\bar{I}_x = (\bar{I}_x)_S + 2(I_x)_{\text{plate}}$$

where

$$(I_x)_{\text{plate}} = \bar{I}_{x_{\text{plate}}} + Ad^2$$
$$= \frac{1}{12}(160 \text{ mm})(20 \text{ mm})^3 + (3200 \text{ mm}^2)[(152.5 + 10) \text{ mm}]^2$$
$$= 84.6067 \times 10^6 \text{ mm}^4$$

Then

$$\bar{I}_x = (90.3 + 2 \times 84.6067) \times 10^6 \text{ mm}^4$$
$$= 259.5134 \times 10^6 \text{ mm}^4 \qquad \text{or} \quad \bar{I}_x = 260 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

$$\bar{k}_x^2 = \frac{\bar{I}_x}{A_{\text{total}}} = \frac{259.5134 \times 10^6 \text{ mm}^4}{12410 \text{ mm}^2} \qquad \text{or} \quad \bar{k}_x = 144.6 \text{ mm} \blacktriangleleft$$

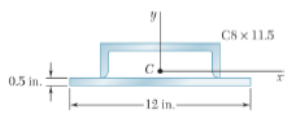
Also

$$\bar{I}_y = (\bar{I}_y)_S + 2(\bar{I}_y)_{\text{plate}}$$
$$= 3.88 \times 10^6 \text{ mm}^4 + 2\left[\frac{1}{12}(20 \text{ mm})(160 \text{ mm})^3\right]$$
$$= 17.5333 \times 10^6 \text{ mm}^4 \qquad \text{or} \quad \bar{I}_y = 17.53 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

and

$$\bar{k}_y^2 = \frac{\bar{I}_y}{A_{\text{total}}} = \frac{17.5333 \times 10^6 \text{ mm}^4}{12{,}410 \text{ mm}^2} \qquad \text{or} \quad \bar{k}_y = 37.6 \text{ mm} \blacktriangleleft$$

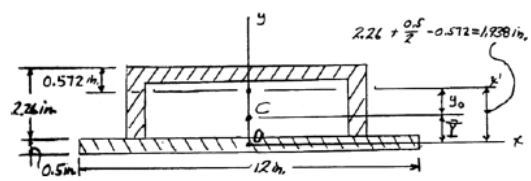
9.53-11y12va



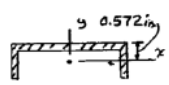
PROBLEM 9.53

A channel and a plate are welded together as shown to form a section that is symmetrical with respect to the y axis. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

SOLUTION



From Figure 9.13B



$B = 3.37 \text{ in}^2$

For C8 × 11.5: $\bar{I}_y = 1.31 \text{ in}^4$

(Note change of axes) $\bar{I}_y = 32.5 \text{ in}^4$

Location of centroid

$$\bar{Y}\Sigma A = \Sigma \bar{y} A: \qquad \bar{Y}[3.7 \text{ in}^2 + (12 \text{ in.})(0.5 \text{ in.})] = (3.37 \text{ in}^2)(1.938 \text{ in.})$$
$$\bar{Y} = 0.69702 \text{ in.}$$

Moment of inertia with respect to x axis.

Plate:

$$I_x = \bar{I}_x + A\bar{Y}^2 = \frac{1}{12}(12 \text{ in.})(0.5 \text{ in.})^3 + (12 \text{ in.})(0.5 \text{ in.})(0.69702 \text{ in.})^2$$
$$= 0.125 + 2.9150 = 3.0400 \text{ in}^4$$

Channel:

$$I_x = \bar{I}_{x'} + A\bar{Y}_0^2 = 1.31 \text{ in}^4 + (3.37 \text{ in.})(1.938 \text{ in.} - 0.69702 \text{ in.})^2$$
$$= 1.31 + 5.1899 = 6.4999 \text{ in}^4$$

Entire section: $I_x = 3.0400 \text{ in}^4 + 6.4999 \text{ in}^4 = 9.5399 \text{ in}^4$ $\qquad \bar{I}_x = 9.54 \text{ in}^4 \blacktriangleleft$

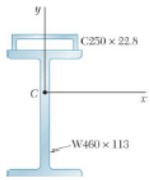
Moment of inertia with respect to y axis.

Plate: $\bar{I}_y = \frac{1}{12}(0.5 \text{ in.})(12 \text{ in.})^3 = 72 \text{ in}^4$

Channel: $\bar{I}_y = 32.5 \text{ in}^4$

Entire section: $\bar{I}_y = 72 \text{ in}^4 + 32.5 \text{ in}^4 = 104.5 \text{ in}^4$ $\qquad \bar{I}_y = 104.5 \text{ in}^4 \blacktriangleleft$

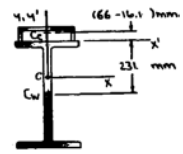
9.54-11y12va



PROBLEM 9.54

The strength of the rolled W section shown is increased by welding a channel to its upper flange. Determine the moments of inertia of the combined section with respect to its centroidal x and y axes.

SOLUTION



W section:

$$A = 14,400 \text{ mm}^2$$
$$\bar{I}_x = 554 \times 10^6 \text{ mm}^4$$
$$\bar{I}_y = 63.3 \times 10^6 \text{ mm}^4$$

Channel:

$$A = 2890 \text{ mm}^2$$
$$\bar{I}_x = 0.945 \times 10^6 \text{ mm}^4$$
$$\bar{I}_y = 28.0 \times 10^6 \text{ mm}^4$$

First locate centroid C of the section.

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
W Section	14,400	−231	−33,26,400
Channel	2,890	49.9	1,44,211
Σ	17,290		−31,82,189

Then $\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\bar{Y}(17,290 \text{ mm}^2) = -3,182,189 \text{ mm}^3$

or $\bar{Y} = -184.047 \text{ mm}$

Now $\bar{I}_x = (I_x)_W + (I_x)_C$

where
$$(I_x)_W = \bar{I}_x + Ad^2$$
$$= 554 \times 10^6 \text{ mm}^4 + (14,400 \text{ mm}^2)(231 - 184.047)^2 \text{ mm}^2$$
$$= 585.75 \times 10^6 \text{ mm}^4$$
$$(I_x)_C = \bar{I}_x - Ad^2$$
$$= 0.945 \times 10^6 \text{ mm}^4 + (2,890 \text{ mm}^2)(49.9 + 184.047)^2 \text{ mm}^2$$
$$= 159.12 \times 10^6 \text{ mm}^4$$

PROBLEM 9.54 (Continued)

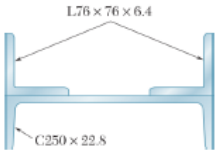
Then $\bar{I}_x = (585.75 + 159.12) \times 10^6 \text{ mm}^4$

or $\bar{I}_x = 745 \times 10^6 \text{ mm}^4$ ◀

also
$$\bar{I}_y = (I_y)_W + (I_y)_C$$
$$= (63.3 + 28.0) \times 10^6 \text{ mm}^4$$

or $\bar{I}_y = 91.3 \times 10^6 \text{ mm}^4$ ◀

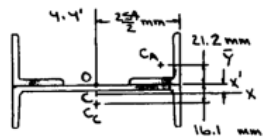
9.55-11y12va



PROBLEM 9.55

Two L76 × 76 × 6.4-mm angles are welded to a C250 × 22.8 channel. Determine the moments of inertia of the combined section with respect to centroidal axes respectively parallel and perpendicular to the web of the channel.

SOLUTION



Angle: $A = 929\ \text{mm}^2$

$\bar{I}_x = \bar{I}_y = 0.512 \times 10^6\ \text{mm}^4$

Channel: $A = 2890\ \text{mm}^2$

$\bar{I}_x = 0.945 \times 10^6\ \text{mm}^4$

$\bar{I}_y = 28.0 \times 10^6\ \text{mm}^4$

First locate centroid C of the section

	A, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
Angle	$2(929) = 1858$	21.2	39,389.6
Channel	2890	−16.1	−46,529
Σ	4748		−7139.4

Then $\bar{Y}\Sigma A = \Sigma \bar{y} A$: $\bar{Y}(4748\ \text{mm}^2) = -7139.4\ \text{mm}^3$

or $\bar{Y} = -1.50366\ \text{mm}$

Now $\bar{I}_x = 2(I_x)_L + (I_x)_C$

where $(I_x)_L = \bar{I}_x + Ad^2 = 0.512 \times 10^6\ \text{mm}^4 + (929\ \text{mm}^2)[(21.2 + 1.50366)\ \text{mm}]^2$

$= 0.990859 \times 10^6\ \text{mm}^4$

$(I_x)_C = \bar{I}_x + Ad^2 = 0.949 \times 10^6\ \text{mm}^4 + (2890\ \text{mm}^2)[(16.1 - 1.50366)\ \text{mm}]^2$

$= 1.56472 \times 10^6\ \text{mm}^4$

Then $\bar{I}_x = [2(0.990859) + 1.56472 \times 10^6]\ \text{mm}^4$

or $\bar{I}_x = 3.55 \times 10^6\ \text{mm}^4$ ◀

PROBLEM 9.55 (Continued)

Also $\bar{I}_y = 2(I_y)_L + (I_y)_C$

where $(I_y)_L = \bar{I}_y + Ad^2 = 0.512 \times 10^6\ \text{mm}^4 + (929\ \text{mm}^2)[(127 - 21.2)\ \text{mm}]^2$

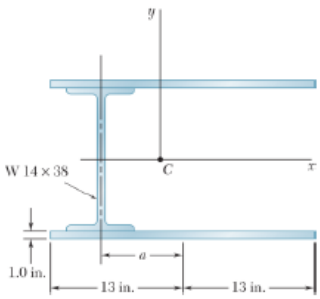
$= 10.9109 \times 10^6\ \text{mm}^4$

$(I_y)_C = \bar{I}_y$

Then $\bar{I}_y = [2(10.9109) + 28.0] \times 10^6\ \text{mm}^4$

or $\bar{I}_y = 49.8 \times 10^6\ \text{mm}^4$ ◀

9.56-11y12va



PROBLEM 9.56

Two steel plates are welded to a rolled W section as indicated. Knowing that the centroidal moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the combined section are equal, determine (*a*) the distance *a*, (*b*) the moments of inertia with respect to the centroidal *x* and *y* axes.

SOLUTION

For W-section: $A = 11.2\text{ in}^2,\quad \bar{I}_x = 385\text{ in}^4,\quad \bar{I}_y = 26.7\text{ in}^4$

Section	Area, in^2	$\bar{x}$ in.	$\bar{x}A$, in^3
Plate	2(1)(26) = 52	a	$52a$
W-section	11.2	0	0
Σ	63.2		$52a$

$$\bar{X}\Sigma A = \Sigma\bar{x}A: \qquad \bar{X}\left(63.2\text{ in}^2\right) = 52a\text{ in}^3$$
$$\bar{x} = 0.82279a\text{ in.}$$

Entire section:

$$\bar{I}_x = (\bar{I}_{x_W} + 2I_{x_{PL}});\quad I_{x_{PL}} = \bar{I}_x + Ad^2$$
$$= \frac{1}{12}(26\text{ in.})(1\text{ in.})^3 + \left(26\text{ in}^3\right)(7.05\text{ in.} + 0.5\text{ in.})^2$$
$$= 1484.232\text{ in}^4$$
$$\bar{I}_x = 385 + 2(1484.232) = 3353.46\text{ in}^4 \qquad \bar{I}_x = 3350\text{ in}^4 \blacktriangleleft$$
$$\bar{I}_x = \bar{I}_y \qquad \bar{I}_y = 3350\text{ in}^4 \blacktriangleleft$$

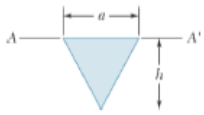
$$\bar{I}_y = (\bar{I}_{y_W} + 2I_{y_{PL}});\quad I_{y_{PL}} = \bar{I}_y + Ad^2$$
$$= \frac{1}{12}(1\text{ in.})(26\text{ in.})^3 + \left(26\text{ in}^2\right)(a - 0.82279a\text{ in.})^2$$
$$= \left(1464.667 + 0.81649a^2\right)\text{ in}^4$$

PROBLEM 9.56 (Continued)

$$\bar{I}_{y_W} = \bar{I}_{y_W} + Ad^2$$
$$= 26.7\text{ in}^4 + \left(11.2\text{ in}^2\right)(0.82279a\text{ in.})^2$$
$$= \left(26.7 + 7.5822a^2\right)\text{in}^4$$
$$\bar{I}_y = \left(26.7 + 7.5822a^2\right) + 2\left(1464.667 + 0.81649a^2\right) = 3353.46$$

$$a = 6.57\text{ in.} \blacktriangleleft$$

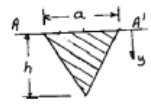
9.57-11y12va



PROBLEM 9.57

The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION



From section 9.2: $R = \gamma \int y dA, \quad M_{AA'} = \gamma \int y^2 dA$

Let y_P= distance of center of pressure from AA'. We must have:

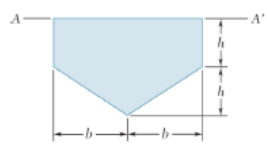
$$R y_P = M_{AA'}: \quad y_P = \frac{M_{AA'}}{R} = \frac{\gamma \int y^2 dA}{\gamma \int y dA} = \frac{I_{AA'}}{\bar{y} A} \qquad (1)$$

For triangular panel:

$$I_{AA'} = \frac{1}{12} a h^3 \qquad \bar{y} = \frac{1}{3} h \qquad A = \frac{1}{2} a h$$

$$y_P = \frac{I_{AA'}}{\bar{y} A} = \frac{\frac{1}{12} a h^3}{\left(\frac{1}{3} h\right)\left(\frac{1}{2} a h\right)} \qquad y_P = \frac{1}{2} h \blacktriangleleft$$

9.58-11y12va

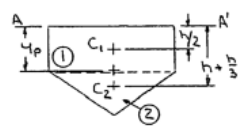


PROBLEM 9.58

The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to Section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

Using the equation developed on page 506 of the text:



$$y_P = \frac{I_{AA'}}{\bar{y} A}$$

Now

$$\bar{y} A = \Sigma \bar{y} A$$

$$= \frac{h}{2}(2b \times h) + \frac{4}{3} h \left(\frac{1}{2} \times 2b \times h\right)$$

$$= \frac{7}{3} b h^2$$

and

$$I_{AA'} = (I_{AA'})_1 + (I_{AA'})_2$$

where

$$(I_{AA'})_1 = \frac{1}{3}(2b)(h)^3 = \frac{2}{3} b h^3$$

$$(I_{AA'})_2 = \bar{I}_x + A d^2 = \frac{1}{36}(2b)(h)^3 + \left(\frac{1}{2} \times 2b \times h\right)\left(\frac{4}{3} h\right)^2$$

$$= \frac{11}{6} b h^3$$

Then

$$I_{AA'} = \frac{2}{3} b h^3 + \frac{11}{6} b h^3 = \frac{5}{2} b h^3$$

Finally,

$$y_P = \frac{\frac{5}{2} b h^3}{\frac{7}{3} b h^2}$$

or $\quad y_P = \frac{15}{14} h \blacktriangleleft$

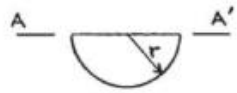
9.59-9y12va

PROBLEM 9.57

The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to Section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

From Section 9.2: $R = \gamma \int y\,dA, \quad M_{AA'} = \gamma \int y^2 dA$



Let y_P = distance of center of pressure from AA'. We must have

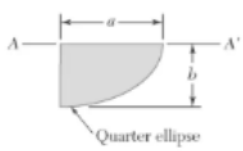
$$Ry_P = M_{AA'}: \quad y_P = \frac{M_{AA'}}{R} = \frac{\gamma \int y^2 dA}{\gamma \int y\,dA} = \frac{I_{AA'}}{\bar{y}A} \qquad (1)$$

For semicircular panel:

$$I_{AA'} = \frac{\pi}{8} r^4 \qquad \bar{y} = \frac{4r}{3\pi} \qquad A = \frac{\pi}{2} r^2$$

$$y_P = \frac{I_{AA'}}{\bar{y}A} = \frac{\frac{\pi}{8} r^4}{\left(\frac{4r}{3\pi}\right)\frac{\pi}{2} r^2} \qquad y_P = \frac{3\pi}{16} r \blacktriangleleft$$

9.60-9y12va

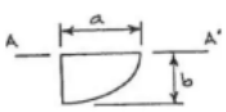


PROBLEM 9.58

The panel shown forms the end of a trough that is filled with water to the line AA'. Referring to Section 9.2, determine the depth of the point of application of the resultant of the hydrostatic forces acting on the panel (the center of pressure).

SOLUTION

See solution of Problem 9.57 for derivation of Eq. (1):



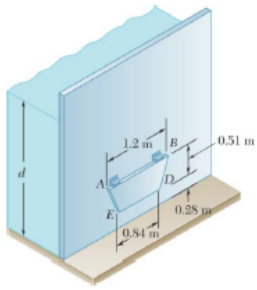
$$y_P = \frac{I_{AA'}}{\bar{y}A} \qquad (1)$$

For quarter ellipse panel:

$$I_{AA'} = \frac{\pi}{16} ab^3 \qquad \bar{y} = \frac{4b}{3\pi} \qquad A = \frac{\pi}{4} ab$$

$$y_P = \frac{I_{AA'}}{\bar{y}A} = \frac{\frac{\pi}{16} ab^3}{\left(\frac{4b}{3\pi}\right)\left(\frac{\pi}{4} ab\right)} \qquad y_P = \frac{3\pi}{16} b \blacktriangleleft$$

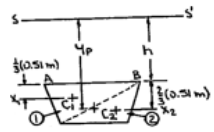
9.61-11y12va



PROBLEM 9.61

A vertical trapezoidal gate that is used as an automatic valve is held shut by two springs attached to hinges located along edge AB. Knowing that each spring exerts a couple of magnitude 1470 N · m, determine the depth d of water for which the gate will open.

SOLUTION



From section 9.2:

$$R = \gamma \bar{y} A \qquad y_P = \frac{I_{SS'}}{\bar{y}A}$$

where R is the resultant of the hydrostatic forces acting on the gate and y_P is the depth to the point of application of R. Now

$$\bar{y}A = \Sigma \bar{y}A = [(h+0.17)\text{ m}]\left(\frac{1}{2}\times 1.2\text{ m}\times 0.51\text{ m}\right) + [(h+0.34)\text{ m}]\left(\frac{1}{2}\times 0.84\times 0.51\text{ m}\right)$$
$$= (0.5202h + 0.124848)\text{ m}^3$$

Recalling that $\gamma = \rho y$, we have

$$R = (10^3\text{ kg/m}^3 \times 9.81\text{ m/s}^2)(0.5202h + 0.124848)\text{ m}^3$$
$$= 5103.162(h+0.24)\text{ N}$$

Also $\qquad I_{SS'} = (I_{SS'})_1 + (I_{SS'})_2$

where

$$(I_{SS'})_1 = I_x + Ad^2$$
$$= \frac{1}{36}(1.2\text{ m})(0.51\text{ m})^3 + \left(\frac{1}{2}\times 1.2\text{ m}\times 0.51\text{ m}\right)[(h+0.17)\text{ m}]^2$$
$$= [0.0044217 + 0.306(h+0.17)^2]\text{ m}^4$$
$$= (0.306h^2 + 0.10404h + 0.0132651)\text{ m}^4$$

$$(I_{SS'})_2 = \bar{I}_{x_2} + Ad^2$$
$$= \frac{1}{36}(0.84\text{ m})(0.51\text{ m})^3 + \left(\frac{1}{2}\times 0.84\text{ m}\times 0.51\text{ m}\right)[(h+0.34)\text{ m}]^2$$
$$= [0.0030952 + 0.2142(h+0.34)^2]\text{ m}^4$$
$$= (0.2142h^2 + 0.145656h + 0.0278567)\text{ m}^4$$

PROBLEM 9.61 (Continued)

Then

$$I_{SS'} = (I_{SS'})_1 + (I_{SS'})_2$$
$$= (0.5202h^2 + 0.249696h + 0.0411218)\text{ m}^4$$

and

$$y_P = \frac{(0.5202h^2 + 0.244696h + 0.0411218)\text{ m}^4}{(0.5202h + 0.124848)\text{ m}^3}$$
$$= \frac{h^2 + 0.48h + 0.07905}{h + 0.24}\text{ m}$$

For the gate to open, require that

$$\Sigma M_{AB}: \quad M_{\text{open}} = (y_P - h)R$$

Substituting

$$2940\text{ N}\cdot\text{m} = \left(\frac{h^2 + 0.48h + 0.07905}{h+0.24} - h\right)\text{m}\times 5103.162(h+0.24)\text{ N}$$

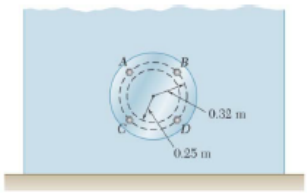
or $\qquad 5103.162(0.24h + 0.07905) = 2940$

or $\qquad h = 2.0711\text{ m}$

Then $\qquad d = (2.0711 + 0.79)\text{ m}$

or $\qquad d = 2.86\text{ m}$ ◀

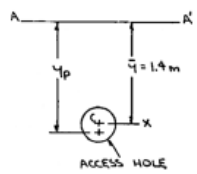
9.62-11y12va



PROBLEM 9.62

The cover for a 0.5-m-diameter access hole in a water storage tank is attached to the tank with four equally spaced bolts as shown. Determine the additional force on each bolt due to the water pressure when the center of the cover is located 1.4 m below the water surface.

SOLUTION



From section 9.2:

$$R = \gamma \bar{y} A \qquad y_P = \frac{I_{AA'}}{\bar{y}A}$$

where R is the resultant of the hydrostatic forces acting on the cover and y_P is the depth to the point of application of R.

Recalling that $\gamma = \rho \cdot g$, we have

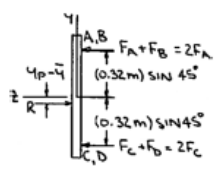
$$R = (10^3 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2)(1.4 \text{ m})[\pi(0.25 \text{ m})^2]$$
$$= 2696.67 \text{ N}$$

Also
$$I_{AA'} = \bar{I}_x + A\bar{y}^2 = \frac{\pi}{4}(0.25 \text{ m})^4 + [\pi(0.25 \text{ m})^2](1.4 \text{ m})^2$$
$$= 0.387913 \text{ m}^4$$

Then
$$y_P = \frac{0.387913 \text{ m}^4}{(1.4 \text{ m})[\pi(0.25 \text{ m})^2]} = 1.41116 \text{ m}$$

Now note that symmetry implies

$$F_A = F_B \qquad F_C = F_D$$



Next consider the free-body of the cover.

We have
$$\circlearrowleft + \Sigma M_{CD} = 0: \quad [2(0.32 \text{ m}) \sin 45°](2F_A) - [0.32 \sin 45° - (1.41116 - 1.4)] \text{ m} \times (2696.67 \text{ N}) = 0$$

or
$$F_A = 640.92 \text{ N}$$

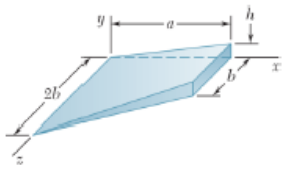
Then
$$\xrightarrow{+} \Sigma F_z = 0: \quad 2(640.92 \text{ N}) + 2F_C - 2696.67 \text{ N} = 0$$

or
$$F_C = 707.42 \text{ N}$$

$$F_A = F_B = 641 \text{ N} \blacktriangleleft$$

and
$$F_C = F_D = 707 \text{ N} \blacktriangleleft$$

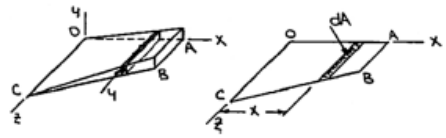
9.63-11y12va



PROBLEM 9.63*

Determine the x coordinate of the centroid of the volume shown. (*Hint:* The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

SOLUTION



First note that $$y = \frac{h}{a}x$$

Now $$\bar{x}\int dV = \int \bar{x}_{EL}\,dV$$

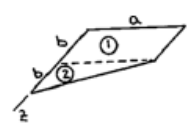
where $$\bar{x}_{EL} = x \quad dV = y\,dA = \left(\frac{h}{a}x\right)dA$$

Then $$\bar{x} = \frac{\int x\left(\frac{h}{a}x\,dA\right)}{\int \frac{h}{a}x\,dA} = \frac{\int x^2\,dA}{\int x\,dA} = \frac{(I_z)_A}{(\bar{x}A)_A}$$

where $(I_z)_A$ and $(\bar{x}_A)_A$ pertain to area.

OABC: $(I_z)_A$ is the moment of inertia of the area with respect to the z axis, $\bar{x}_A$ is the x coordinate of the centroid of the area, and A is the area of *OABC*. Then

$$\begin{aligned}(I_z)_A &= (I_z)_{A_1} + (I_z)_{A_2}\\ &= \frac{1}{3}(b)(a)^3 + \frac{1}{12}(b)(a)^3\\ &= \frac{5}{12}a^3b\end{aligned}$$



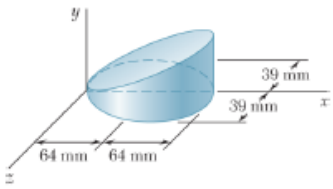
and $$\begin{aligned}(\bar{x}A)_A &= \Sigma\bar{x}A\\ &= \left[\left(\frac{a}{2}\right)(a\times b)\right] + \left[\left(\frac{a}{3}\right)\left(\frac{1}{2}\times a\times b\right)\right]\\ &= \frac{2}{3}a^2b\end{aligned}$$

PROBLEM 9.63* (Continued)

Finally, $$\bar{x} = \frac{\frac{5}{12}a^3b}{\frac{2}{3}a^2b}$$

or $$\bar{x} = \frac{5}{8}a \blacktriangleleft$$

9.64-11y12va



PROBLEM 9.64*

Determine the x coordinate of the centroid of the volume shown; this volume was obtained by intersecting an elliptic cylinder with an oblique plane. (*Hint:* The height y of the volume is proportional to the x coordinate; consider an analogy between this height and the water pressure on a submerged surface.)

SOLUTION

Following the "Hint," it can be shown that (see solution to Problem 9.63)

$$\bar{x} = \frac{(I_z)_A}{(\bar{x}A)_A} x$$

where $(I_z)_A$ and $(\bar{x}A)_A$ are the moment of inertia and the first moment of the area, respectively, of the elliptical area of the base of the volume with respect to the z axis. Then

$$\begin{aligned}(I_z)_A &= \bar{I}_z + Ad^2 \\ &= \frac{\pi}{4}(39\text{ mm})(64\text{ mm})^3 + [\pi(64\text{ mm})(39\text{ mm})](64\text{ mm})^2 \\ &= 12.779520\pi \times 10^6\text{ mm}^4\end{aligned}$$

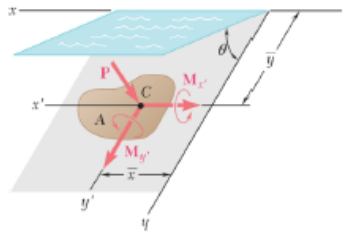
$$\begin{aligned}(\bar{x}A)_A &= (64\text{ mm})[\pi(64\text{ mm})(39\text{ mm})] \\ &= 0.159744\pi \times 10^6\text{ mm}^3\end{aligned}$$

Finally

$$\bar{x} = \frac{12.779520\pi \times 10^6\text{ mm}^4}{0.159744\pi \times 10^6\text{ mm}^4}$$

or $\bar{x} = 80.0\text{ mm}$ ◀

9.65-11y12va



PROBLEM 9.65*

Show that the system of hydrostatic forces acting on a submerged plane area A can be reduced to a force $\mathbf{P}$ at the centroid C of the area and two couples. The force $\mathbf{P}$ is perpendicular to the area and is of magnitude $P = \gamma A\bar{y}\sin\theta$, where γ is the specific weight of the liquid, and the couples are $\mathbf{M}_{x'} = (\gamma\bar{I}_{x'}\sin\theta)\mathbf{i}$ and $\mathbf{M}_{y'} = (\gamma\bar{I}_{x'y'}\sin\theta)\mathbf{j}$, where $\bar{I}_{x'y'} = \int x'y'\,dA$ (see section 9.8). Note that the couples are independent of the depth at which the area is submerged.

SOLUTION

The pressure p at an arbitrary depth $(y\sin\theta)$ is

$$p = \gamma(y\sin\theta)$$

so that the hydrostatic force dF exerted on an infinitesimal area dA is

$$dF = (\gamma y\sin\theta)dA$$

Equivalence of the force $\mathbf{P}$ and the system of infinitesimal forces dF requires

$$\Sigma F: \quad P = \int dF = \int \gamma y\sin\theta\,dA = \gamma\sin\theta\int y\,dA$$

or $\quad P = \gamma A\bar{y}\sin\theta$ ◀

Equivalence of the force and couple $(\mathbf{P},\ \mathbf{M}_{x'} + \mathbf{M}_{y'})$ and the system of infinitesimal hydrostatic forces requires

$$\Sigma M_x: \quad -\bar{y}P - M_{x'} = \int(-y\,dF)$$

Now

$$-\int y\,dF = -\int y(\gamma y\sin\theta)dA = -\gamma\sin\theta\int y^2 dA$$

$$= -(\gamma\sin\theta)I_x$$

Then

$$-\bar{y}P - M_{x'} = -(\gamma\sin\theta)I_x$$

or

$$M_{x'} = (\gamma\sin\theta)I_x - \bar{y}(\gamma A\bar{y}\sin\theta)$$

$$= \gamma\sin\theta(I_x - A\bar{y}^2)$$

or $\quad M_{x'} = \gamma\bar{I}_{x'}\sin\theta$ ◀

$$\Sigma M_y: \quad \bar{x}P + M_{y'} = \int x\,dF$$

Now

$$\int x\,dF = \int x(\gamma y\sin\theta)dA = \gamma\sin\theta\int xy\,dA$$

$$= (\gamma\sin\theta)I_{xy} \qquad \text{(Equation 9.12)}$$

PROBLEM 9.65* (Continued)

Then

$$\bar{x}P + M_{y'} = (\gamma\sin\theta)I_{xy}$$

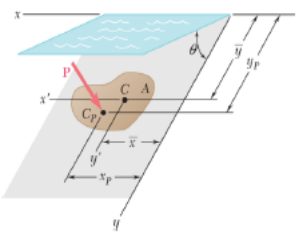
or

$$M_{y'} = (\gamma\sin\theta)I_{xy} - \bar{x}(\gamma A\bar{y}\sin\theta)$$

$$= \gamma\sin\theta(I_{xy} - A\bar{x}\,\bar{y})$$

or $\quad M_{y'} = \gamma\bar{I}_{x'y'}\sin\theta$ ◀

9.66-11y12va



PROBLEM 9.66*

Show that the resultant of the hydrostatic forces acting on a submerged plane area A is a force **P** perpendicular to the area and of magnitude $P = \gamma A\bar{y}\sin\theta = \bar{p}A$, where γ is the specific weight of the liquid and $\bar{p}$ is the pressure at the centroid C of the area. Show that **P** is applied at a Point C_P, called the center of pressure, whose coordinates are $x_P = I_{xy}/A\bar{y}$ and $y_P = I_x/A\bar{y}$, where $\bar{I}_{xy} = \int xydA$ (see section 9.8). Show also that the difference of ordinates $y_P - \bar{y}$ is equal to $\bar{k}_x^2/\bar{y}$ and thus depends upon the depth at which the area is submerged.

SOLUTION

The pressure P at an arbitrary depth $(y\sin\theta)$ is

$$P = \gamma(y\sin\theta)$$

so that the hydrostatic force dP exerted on an infinitesimal area dA is

$$dP = (\gamma y\sin\theta)dA$$

The magnitude **P** of the resultant force acting on the plane area is then

$$P = \int dP = \int \gamma y\sin\theta\, dA = \gamma\sin\theta\int ydA$$
$$= \gamma\sin\theta(\bar{y}A)$$

Now $\bar{p} = \gamma\bar{y}\sin\theta$ $\qquad P = \bar{p}A$ ◀

Next observe that the resultant **P** is equivalent to the system of infinitesimal forces d**P**. Equivalence then requires

$$\Sigma M_x:\quad -y_P P = -\int ydP$$

Now
$$\int ydP = \int y(\gamma y\sin\theta)dA = \gamma\sin\theta\int y^2 dA$$
$$= (\gamma\sin\theta)I_x$$

Then
$$y_P P = (\gamma\sin\theta)I_x$$

or
$$y_P = \frac{(\gamma\sin\theta)I_x}{\gamma\sin\theta(\bar{y}A)}$$

or $\quad y_P = \dfrac{I_x}{A\bar{y}}$ ◀

PROBLEM 9.66* (Continued)

$$\Sigma M_y:\quad x_P P = \int xdP$$

Now
$$\int xdP = \int x(\gamma y\sin\theta)dA = \gamma\sin\theta\int xydA$$
$$= (\gamma\sin\theta)I_{xy} \qquad \text{(Equation 9.12)}$$

Then
$$x_P P = (\gamma\sin\theta)I_{xy}$$

or
$$x_P = \frac{(\gamma\sin\theta)I_{xy}}{\gamma\sin\theta(\bar{y}A)}$$

or $\quad x_P = \dfrac{I_{xy}}{A\bar{y}}$ ◀

Now $\quad I_x = \bar{I}_{x'} + A\bar{y}^2$

From above $\quad I_x = (A\bar{y})y_P$

By definition $\quad \bar{I}_{x'} = \bar{k}_{x'}^2 A$

Substituting $\quad (A\bar{y})y_P = \bar{k}_{x'}^2 A + A\bar{y}^2$

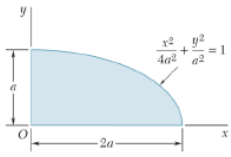
Rearranging yields $\qquad y_P - \bar{y} = \dfrac{\bar{k}_{x'}^2}{\bar{y}}$ ◀

Although $\bar{k}_{x'}$ is not a function of the depth of the area (it depends only on the shape of A), $\bar{y}$ is dependent on the depth.

$$(y_P - \bar{y}) = f(\text{depth})$$

Transformación de momentos de inercia

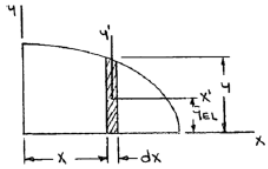
9.67-11y12va



PROBLEM 9.67

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION



First note $\quad y = a\sqrt{1-\frac{x^2}{4a^2}}$

$$= \frac{1}{2}\sqrt{4a^2 - x^2}$$

We have $\quad dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}dA$

where $\quad d\bar{I}_{x'y'} = 0 \quad$ (symmetry) $\quad \bar{x}_{EL} = x$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{4}\sqrt{4a^2 - x^2}$$

$$dA = ydx = \frac{1}{2}\sqrt{4a^2 - x^2dx}$$

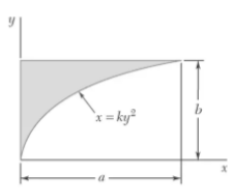
Then $\quad I_{xy} = \int dI_{xy} = \int_0^{2a} x\left(\frac{1}{4}\sqrt{4a^2 - x^2}\right)\left(\frac{1}{2}\sqrt{4a^2 - x^2}\right)dx$

$$= \frac{1}{8}\int_0^{2a}(4a^2x - x^3)dx = \frac{1}{8}\left[2a^2x^2 - \frac{1}{4}x^4\right]_0^{2a}$$

$$= \frac{a^4}{8}\left[2(2)^2 - \frac{1}{4}(2)^4\right]$$

or $\quad I_{xy} = \frac{1}{2}a^4$ ◀

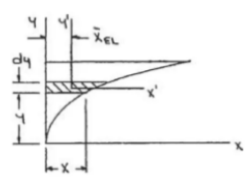
9.69-9y12va



PROBLEM 9.68

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION



At $\quad x = a,\ y = b: \quad a = kb^2$

or $\quad k = \frac{a}{b^2}$

Then $\quad x = \frac{a}{b^2}y^2$

We have $\quad dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}dA$

where $\quad d\bar{I}_{x'y'} = 0 \quad$ (symmetry) $\quad \bar{x}_{EL} = \frac{1}{2}x = \frac{a}{2b^2}y^2$

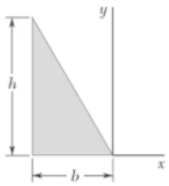
$$\bar{y}_{EL} = y \quad dA = xdy = \frac{a}{b^2}y^2dy$$

Then $\quad I_{xy} = \int dI_{xy} = \int_0^b\left(\frac{a}{2b^2}y^2\right)(y)\left(\frac{a}{b^2}y^2dy\right)$

$$= \frac{a^2}{2b^4}\int_0^b y^5dy = \frac{a^2}{2b^4}\left[\frac{1}{6}y^6\right]_0^b$$

or $\quad I_{xy} = \frac{1}{12}a^2b^2$ ◀

9.70-9y12va



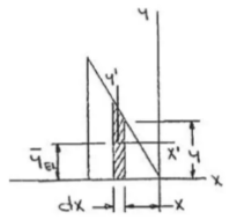
PROBLEM 9.69

Determine by direct integration the product of inertia of the given area with respect to the x and y axes.

SOLUTION

First note that

$$y = -\frac{h}{b}x$$



Now $\quad dI_{xy} = d\bar{I}_{x'y'} + \bar{x}_{EL}\bar{y}_{EL}dA$

where $\quad d\bar{I}_{x'y'} = 0$ (symmetry)

$$\bar{x}_{EL} = x \quad \bar{y}_{EL} = \frac{1}{2}y = -\frac{1}{2}\frac{h}{b}x$$

$$dA = y\,dx = -\frac{h}{b}x\,dx$$

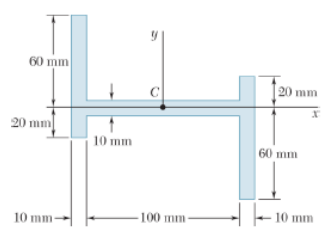
Then

$$I_{xy} = \int dI_{xy} = \int_{-b}^{0} x\left(-\frac{1}{2}\frac{h}{b}x\right)\left(-\frac{h}{b}x\,dx\right)$$

$$= \frac{1}{2}\frac{h^2}{b^2}\int_{-b}^{0} x^3 dx = \frac{1}{2}\frac{h^2}{b^2}\left[\frac{1}{4}x^4\right]_{-b}^{0}$$

or $\quad I_{xy} = -\frac{1}{8}b^2h^2$ ◀

9.71-9,11y12va

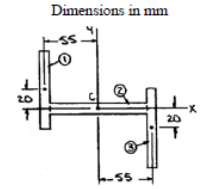


PROBLEM 9.71

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION

Dimensions in mm



We have $\quad \bar{I}_{xy} = (I_{xy})_1 + (\bar{I}_{xy})_2 + (I_{xy})_3$

Now symmetry implies

$$(\bar{I}_{xy})_2 = 0$$

and for the other rectangles

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

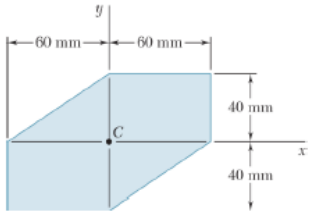
where $\quad I_{x'y'} = 0$ (symmetry)

Thus $\quad \bar{I}_{xy} = (\bar{x}\bar{y}A)_1 + (\bar{x}\bar{y}A)_3$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\bar{y}A$, mm^4
1	$10 \times 80 = 800$	−55	20	−880,000
3	$10 \times 80 = 800$	55	−20	−880,000
Σ				−1,760,000

$\bar{I}_{xy} = -1.760 \times 10^6\ \text{mm}^4$ ◀

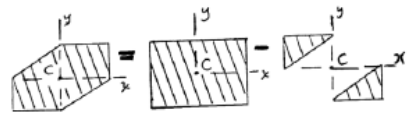
9.72-11y12va



PROBLEM 9.72

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

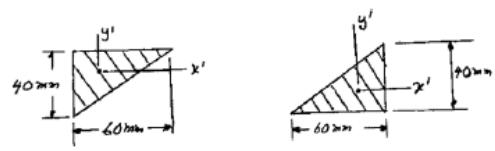
SOLUTION



Given area = Rectangle − (Two triangles)

For rectangle $I_{xy} = 0$, by symmetry

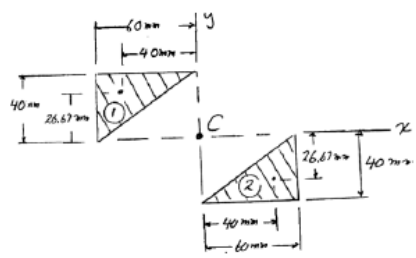
For each triangle:



Compare these triangles with the triangle of sample Problem 9.6, where $\bar{I}_{x'y'} = -\frac{1}{72}b^2h^2$.

For orientation of axes of this problem,

$$\bar{I}_{x'y'} = +\frac{1}{72}b^2h^2 = \frac{1}{72}(60\text{ mm})^2(40\text{ mm})^2 = +80\times10^3\text{ mm}^4$$



PROBLEM 9.72 (Continued)

	Area, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\bar{y}A$, mm^4
1	$\frac{1}{2}(60)(40) = 1200$	−40	+26.67	-1.280×10^6
2	$\frac{1}{2}(60)(40) = 1200$	+40	−26.67	-1.280×10^6

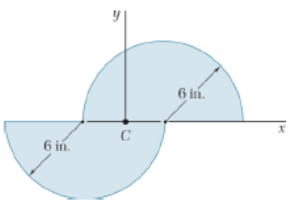
$$\Sigma\,\bar{x}\,\bar{y}\,A = -2.56\times10^6\text{ mm}^4$$

For two triangles:

$$I_{xy} = \Sigma(\bar{I}_{x'y'} + \bar{x}\,\bar{y}\,A) = +2(80\times10^3) - 2.56\times10^6$$
$$= -2.40\times10^6\text{ mm}^4$$

Since we must subtract triangles, $I_{xy} = +2.40\times10^6\text{ mm}^4$ ◀

9.73-9,11y12va



PROBLEM 9.73

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

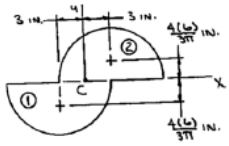
SOLUTION

We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

For each semicircle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

and $I_{x'y'} = 0$ (symmetry)

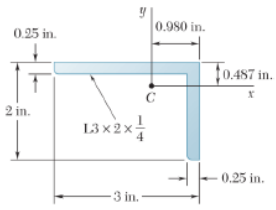
Thus $\bar{I}_{xy} = \Sigma\,\bar{x}\,\bar{y}A$



	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\bar{y}A$
1	$\frac{\pi}{2}(6)^2 = 18\pi$	−3	$-\frac{8}{\pi}$	432
2	$\frac{\pi}{2}(6)^2 = 18\pi$	3	$\frac{8}{\pi}$	432
Σ				864

$\bar{I}_{xy} = 864 \text{ in}^4$ ◀

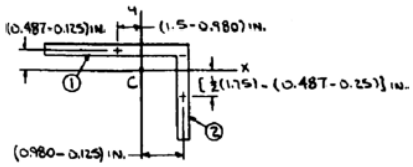
9,74-11y12va



PROBLEM 9.74

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



We have $\bar{I}_{xy} = (I_{xy})_1 + (I_{xy})_2$

For each rectangle

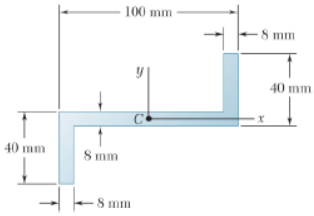
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

and $\bar{I}_{x'y'} = 0$ (symmetry)

Thus $\bar{I}_{xy} = \Sigma\,\bar{x}\,\bar{y}A$

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
1	$3 \times 0.25 = 0.75$	−0.520	0.362	−0.141180
2	$0.25 \times 1.75 = 0.4375$	0.855	−0.638	−0.238652
Σ				−0.379832

$\bar{I}_{xy} = -0.380 \text{ in}^4$ ◀



PROBLEM 9.75

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

9.75-11y12va

SOLUTION

We have $\quad I_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$

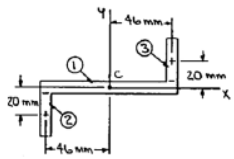
Now symmetry implies $\quad (\bar{I}_{xy})_1 = 0$

and for the other rectangles

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

where $\quad \bar{I}_{x'y'} = 0$ (symmetry)

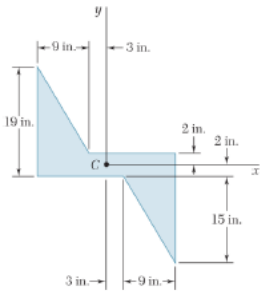
Thus $\quad \bar{I}_{xy} = (\bar{x}\,\bar{y}A)_2 + (\bar{x}\,\bar{y}A)_3$



	A, mm^2	$\bar{X}$, mm	$\bar{y}$, mm	$\bar{x}\,\bar{y}A$, mm^4
2	$8 \times 32 = 256$	–46	–20	235,520
3	$8 \times 32 = 256$	46	20	235,520
Σ				471,040

$\bar{I}_{xy} = 471 \times 10^3$ mm^4 ◀

9.76-11y12va



PROBLEM 9.76

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

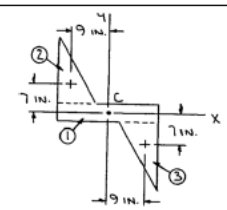
SOLUTION

We have $\quad \bar{I}_{xy} = (\bar{I}_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$

Now, symmetry implies

$$(\bar{I}_{xy})_1 = 0$$

and for each triangle $\quad I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$



where, using the results of Sample Problem 9.6, $\bar{I}_{x'y'} = -\frac{1}{72}b^2h^2$ for both triangles. Note that the sign of $\bar{I}_{x'y'}$ is unchanged because the angles of rotation are 0° and 180° for triangles 2 and 3, respectively.

Now

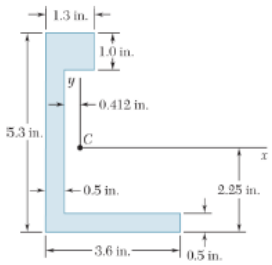
	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
2	$\frac{1}{2}(9)(15) = 67.5$	–9	7	–4252.5
3	$\frac{1}{2}(9)(15) = 67.5$	9	–7	–4252.5
Σ				–8505

Then $\quad \bar{I}_{xy} = 2\left[-\frac{1}{72}(9\text{ in.})^2\,(15\text{ in.})^2\right] - 8505\text{ in}^4$

$$= -9011.25\text{ in}^4$$

or $\quad \bar{I}_{xy} = -9010$ in^4 ◀

9.77-11y12va



PROBLEM 9.77

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

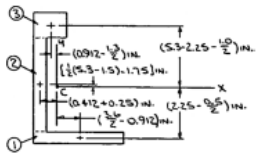
SOLUTION

We have $\bar{I}_{xy} = (\bar{I}_{xy})_1 + (\bar{I}_{xy})_2 + (\bar{I}_{xy})_3$

For each rectangle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

and $\bar{I}_{x'y'} = 0$ (symmetry)

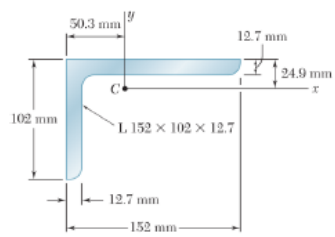
Thus $I_{xy} = \Sigma \bar{x}\,\bar{y}A$



	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
1	$3.6 \times 0.5 = 1.8$	0.888	−2.00	−3.196
2	$0.5 \times 3.8 = 1.9$	−0.662	0.15	−0.18867
3	$1.3 \times 1.0 = 1.3$	−0.262	2.55	−0.86853
Σ				−4.25320

$\bar{I}_{xy} = -4.25 \text{ in}^4$ ◀

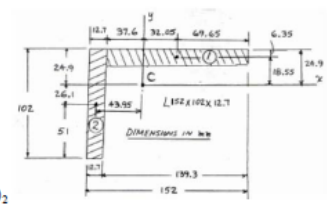
9.78-11y12va



PROBLEM 9.78

Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



We have $\bar{I}_{xy} = (\bar{I}_{xy})_1 + (\bar{I}_{xy})_2$

For each rectangle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$

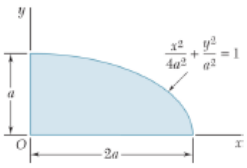
and $\bar{I}_{x'y'} = 0$ (symmetry)

Thus $I_{xy} = \Sigma \bar{x}\,\bar{y}A$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\,\bar{y}A$, mm^4
1	$139.3 \times 12.7 = 1769.11$	+32.05	+18.55	$+1051.78 \times 10^3$
2	$12.7 \times (102) = 1295.4$	-43.95	-26.1	$+1485.95 \times 10^3$
Σ				2537.7×10^3

$\bar{I}_{xy} = 2.54 \times 10^6 \text{ mm}^4$ ◀

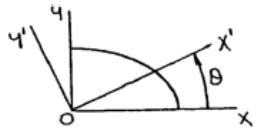
9.79-11y12va



PROBLEM 9.79

Determine for the quarter ellipse of Problem 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the x and y axes about O (a) through 45° counterclockwise, (b) through 30° clockwise.

SOLUTION



From Figure 9.12:

$$I_x = \frac{\pi}{16}(2a)(a)^3 = \frac{\pi}{8}a^4$$

$$I_y = \frac{\pi}{16}(2a)^3(a) = \frac{\pi}{2}a^4$$

From Problem 9.67:

$$I_{xy} = \frac{1}{2}a^4$$

First note

$$\frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = \frac{5}{16}\pi a^4$$

$$\frac{1}{2}(I_x - I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right) = -\frac{3}{16}\pi a^4$$

Now use Equations (9.18), (9.19), and (9.20).

Equation (9.18):

$$I_{x'} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\cos 2\theta - I_{xy}\sin 2\theta$$

$$= \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos 2\theta - \frac{1}{2}a^4 \sin 2\theta$$

Equation (9.19):

$$I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta + I_{xy}\sin 2\theta$$

$$= \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4 \cos 2\theta + \frac{1}{2}a^4 \sin 2\theta$$

Equation (9.20):

$$I_{x'y'} = \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta$$

$$= -\frac{3}{16}\pi a^4 \sin 2\theta + \frac{1}{2}a^4 \cos 2\theta$$

PROBLEM 9.79 (Continued)

(a) $\theta = +45°$:

$$I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos 90° - \frac{1}{2}a^4 \sin 90°$$

or $I_{x'} = 0.482a^4$ ◀

$$I_{y'} = \frac{5}{16}\pi + \frac{3}{16}\pi a^4 \cos 90° + \frac{1}{2}a^4$$

or $I_{y'} = 1.482a^4$ ◀

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin 90° + \frac{1}{2}a^4 \cos 90°$$

or $I_{x'y'} = -0.589a^4$ ◀

(b) $\theta = -30°$:

$$I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos(-60°) - \frac{1}{2}a^4 \sin(-60°)$$

or $I_{x'} = 1.120a^4$ ◀

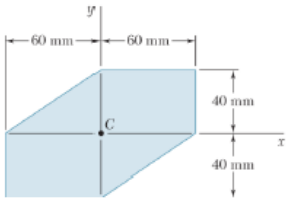
$$I_{y'} = \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4 \cos(-60°) + \frac{1}{2}a^4 \sin(-60°)$$

or $I_{y'} = 0.843a^4$ ◀

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin(-60°) + \frac{1}{2}a^4 \cos(-60°)$$

or $I_{x'y'} = 0.760a^4$ ◀

9.80-11y12va

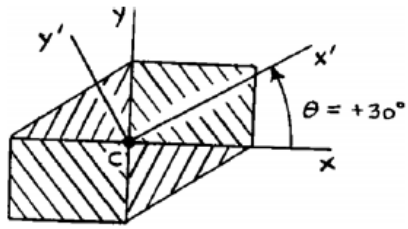


PROBLEM 9.80

Determine the moments of inertia and the product of inertia of the area of Problem 9.72 with respect to new centroidal axes obtained by rotating the x and y axes 30° counterclockwise.

SOLUTION

From Problem 9.72: $\bar{I}_{xy} = 2.40 \times 10^6 \text{ mm}^4$



Now, with two rectangles and two triangles:

$$\bar{I}_x = 2\left[\frac{1}{3}(60 \text{ mm})(40 \text{ mm})^3\right] + 2\left[\frac{1}{12}(60 \text{ mm})(40 \text{ mm})^3\right]$$

$$\bar{I}_x = 3.20 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 2\left[\frac{1}{3}(40 \text{ mm})(60 \text{ mm})^3\right] + 2\left[\frac{1}{12}(40 \text{ mm})(60 \text{ mm})^3\right] = 7.20 \times 10^6 \text{ mm}^4$$

Now

$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(3.20 + 7.20) \times 10^6 = 5.20 \times 10^6 \text{ mm}^4$$

$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(3.20 - 7.20) \times 10^6 = -2.00 \times 10^6 \text{ mm}^4$$

Using Eqs. (9.18), (9.19), (9.20):

Eq. (9.18):

$$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$

$$= [5.20 + (-2.00)\cos 60° - (2.40)\sin 60°] \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'} = 2.12 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.80 (Continued)

Eq. (9.19):

$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= [5.20 - (-2.00)\cos 60° + (2.40)\sin 60°] \times 10^6 \text{ mm}^4$$

or $\bar{I}_{y'} = 8.28 \times 10^6 \text{ mm}^4$ ◀

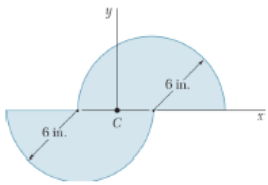
Eq. (9.20):

$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= [(-2.00)\sin 60° + (2.40)\cos 60°] \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'y'} = -0.532 \times 10^6 \text{ mm}^4$ ◀

9.81-11y12va



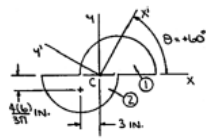
PROBLEM 9.81

Determine the moments of inertia and the product of inertia of the area of Problem 9.73 with respect to new centroidal axes obtained by rotating the x and y axes 60° counterclockwise.

SOLUTION

From Problem 9.73:

$$\bar{I}_{xy} = 864 \text{ in}^4$$



Now $\quad \bar{I}_x = (I_x)_1 + (I_x)_2$

where $\quad (I_x)_1 = (I_x)_2 = \frac{\pi}{8}(6 \text{ in.})^4$

$$= 162\pi \text{ in}^4$$

Then $\quad \bar{I}_x = 2(162\pi \text{ in}^4) = 324\pi \text{ in}^4$

Also $\quad \bar{I}_y = (I_y)_1 + (I_y)_2$

where $\quad (I_y)_1 = (I_y)_2 = \frac{\pi}{8}(6 \text{ in.})^4 + \left[\frac{\pi}{2}(6 \text{ in.})^2\right](3 \text{ in.})^2 = 324\pi \text{ in}^4$

Then $\quad \bar{I}_y = 2(324\pi \text{ in}^4) = 648\pi \text{ in}^4$

Now $\quad \frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(324\pi + 648\pi) = 486\pi \text{ in}^4$

$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(324\pi - 648\pi) = -162\pi \text{ in}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18): $\quad \bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$

$$= [486\pi + (-162\pi)\cos 120° - 864 \sin 120°] \text{ in}^4$$

or $\quad \bar{I}_{x'} = 1033 \text{ in}^4$ ◀

PROBLEM 9.81 (Continued)

Eq. (9.19): $\quad \bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$

$$= [486\pi - (-162\pi)\cos 120° + 864 \sin 120°] \text{ in}^4$$

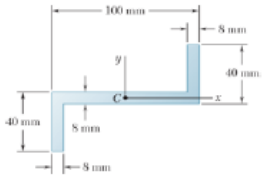
or $\quad \bar{I}_{y'} = 2020 \text{ in}^4$ ◀

Eq. (9.20): $\quad \bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$

$$= [(-162\pi)\sin 120° + 864 \cos 120°] \text{ in}^4$$

or $\quad \bar{I}_{x'y'} = -873 \text{ in}^4$ ◀

9.82-11y2va



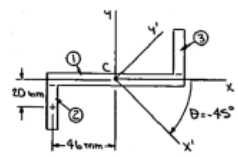
PROBLEM 9.82

Determine the moments of inertia and the product of inertia of the area of Problem 9.75 with respect to new centroidal axes obtained by rotating the x and y axes 45° clockwise.

SOLUTION

From Problem 9.75:

$$\bar{I}_{xy} = 471{,}040 \text{ mm}^4$$



Now $\bar{I}_x = (\bar{I}_x)_1 + (I_x)_2 + (I_x)_3$

where $(\bar{I}_x)_1 = \frac{1}{12}(100 \text{ mm})(8 \text{ mm})^3$

$= 4266.67 \text{ mm}^4$

$$(I_x)_2 = (I_x)_3 = \frac{1}{12}(8 \text{ mm})(32 \text{ mm})^3 + [(8 \text{ mm})(32 \text{ mm})](20 \text{ mm})^2$$

$$= 124{,}245.33$$

Then $\bar{I}_x = [4266.67 + 2(124{,}245.33)] \text{ mm}^4 = 252{,}757 \text{ mm}^4$

Also $\bar{I}_y = (\bar{I}_y)_1 + (I_y)_2 + (I_y)_3$

where $(\bar{I}_y)_1 = \frac{1}{12}(8 \text{ mm})(100 \text{ mm})^3 = 666{,}666.7 \text{ mm}^4$

$$(\bar{I}_y)_2 = (I_y)_3 = \frac{1}{12}(32 \text{ mm})(8 \text{ mm})^3 + [(8 \text{ mm})(32 \text{ mm})](46 \text{ mm})^2$$

$$= 543{,}061.3 \text{ mm}^4$$

Then $\bar{I}_y = [666{,}666.7 + 2(543{,}061.3)] \text{ mm}^4 = 1{,}752{,}789 \text{ mm}^4$

Now $\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(252{,}757 + 1{,}752{,}789) \text{ mm}^4 = 1{,}002{,}773 \text{ mm}^4$

$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(252{,}757 - 1{,}752{,}789) \text{ mm}^4 = -750{,}016 \text{ mm}^4$$

Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18): $\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$

$$= [1{,}002{,}773 + (-750{,}016)\cos(-90°) - 471{,}040\sin(-90°)]$$

or $\bar{I}_{x'} = 1.474 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.82 (Continued)

Eq. (9.19): $\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$

$$= [1{,}002{,}773 - (-750{,}016)\cos(-90°) + 471{,}040\sin(-90°)]$$

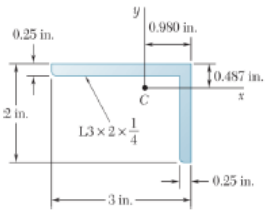
or $\bar{I}_{y'} = 0.532 \times 10^6 \text{ mm}^4$ ◀

Eq. (9.20): $\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$

$$= [(-750{,}016)\sin(-90°) + 471{,}040\cos(-90°)]$$

or $\bar{I}_{x'y'} = 0.750 \times 10^6 \text{ mm}^4$ ◀

9.83-11y12va

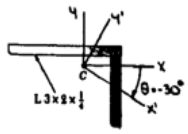


PROBLEM 9.83

Determine the moments of inertia and the product of inertia of the $L3 \times 2 \times \frac{1}{4}$-in. angle cross section of Problem 9.74 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

SOLUTION

From Figure 9.13:



$$\bar{I}_x = 0.390 \text{ in}^4$$
$$\bar{I}_y = 1.09 \text{ in}^4$$

From Problem 9.74:

$$\bar{I}_{xy} = -0.37983 \text{ in}^4$$

Now
$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(0.390 + 1.09) \text{ in}^4 = 0.740 \text{ in}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(0.390 - 1.09) \text{ in}^4 = -0.350 \text{ in}^4$$

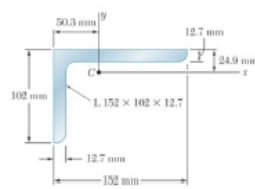
Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):
$$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [0.740 + (-0.350)\cos(-60°) - (-0.37983)\sin(-60°)]$$
or $\bar{I}_{x'} = 0.236 \text{ in}^4$ ◀

Eq. (9.19):
$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$
$$= [0.740 - (-0.350)\cos(-60°) + (-0.37983)\sin(-60°)]$$
or $\bar{I}_{y'} = 1.244 \text{ in}^4$ ◀

Eq. (9.20):
$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$
$$= [(-0.350)\sin(-60°) + (-0.37983)\cos(-60°)]$$
or $I_{x'y'} = 0.1132 \text{ in}^4$ ◀

9.84-11y12va



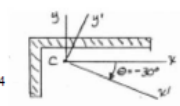
PROBLEM 9.84

Determine the moments of inertia and the product of inertia of the L152 × 102 × 12.7-mm angle cross section of Prob. 9.78 with respect to new centroidal axes obtained by rotating the x and y axes 30° clockwise.

SOLUTION

From Figure 9.13:

Note the axes are reversed in sketch.



$$\bar{I}_x = 2.59 \times 10^6 \text{ mm}^4$$
$$\bar{I}_y = 7.20 \times 10^6 \text{ mm}^4$$

From Problem 9.78:

$$\bar{I}_{xy} = 2.54 \times 10^6 \text{ mm}^4$$

Now
$$\frac{1}{2}(\bar{I}_x + \bar{I}_y) = \frac{1}{2}(2.59 + 7.20) \times 10^6 \text{ mm}^4$$
$$= 4.895 \times 10^6 \text{ mm}^4$$
$$\frac{1}{2}(\bar{I}_x - \bar{I}_y) = \frac{1}{2}(2.59 - 7.20) \times 10^6 \text{ mm}^4 = -2.305 \times 10^6 \text{ mm}^4$$

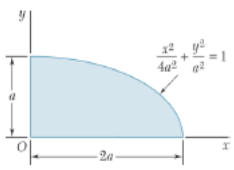
Using Eqs. (9.18), (9.19), and (9.20):

Eq. (9.18):
$$\bar{I}_{x'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) + \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$
$$= [4.895 - 2.305\cos(-60°) - 2.54\sin(-60°)] \times 10^6 \text{ mm}^4$$
or $\bar{I}_{x'} = 5.94 \times 10^6 \text{ mm}^4$ ◀

Eq. (9.19):
$$\bar{I}_{y'} = \frac{1}{2}(\bar{I}_x + \bar{I}_y) - \frac{1}{2}(\bar{I}_x - \bar{I}_y)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$
$$= [4.895 + 2.305\cos(-60°) + 2.54\sin(-60°)] \times 10^6 \text{ mm}^4$$
or $\bar{I}_{y'} = 3.85 \times 10^6 \text{ mm}^4$ ◀

Eq. (9.20):
$$\bar{I}_{x'y'} = \frac{1}{2}(\bar{I}_x - \bar{I}_y)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$
$$= [(-2.305\sin(-60°) + 2.54\cos(-60°)] \times 10^6 \text{ mm}^4$$
or $\bar{I}_{x'y'} = 3.27 \times 10^6 \text{ mm}^4$ ◀

9.85-11y12va



PROBLEM 9.85

For the quarter ellipse of Problem 9.67, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

SOLUTION

From Problem 9.79: $I_x = \frac{\pi}{8}a^4 \qquad I_y = \frac{\pi}{2}a^4$

Problem 9.67: $I_{xy} = \frac{1}{2}a^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4}$$

$$= \frac{8}{3\pi} = 0.84883$$

Then $2\theta_m = 40.326° \quad \text{and} \quad 220.326°$

or $\theta_m = 20.2° \text{ and } 110.2°$ ◀

Also Eq. (9.27):

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right)$$

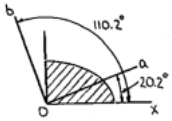
$$\pm\sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2}$$

$$= (0.981,748 \pm 0.772,644)a^4$$

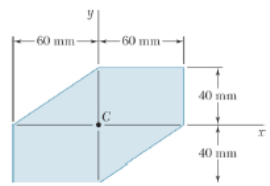
or $I_{\max} = 1.754a^4$ ◀

and $I_{\min} = 0.209a^4$ ◀

By inspection, the a axis corresponds to $I_{\min}$ and the b axis corresponds to $I_{\max}$.



9.86-11y12va



PROBLEM 9.86

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.72.

SOLUTION

From Problem 9.80: $\bar{I}_x = 3.20\times10^6\,\text{mm}^4$

$\bar{I}_y = 7.20\times10^6\,\text{mm}^4$

From Problem 9.72: $\bar{I}_{xy} = 2.40\times10^6\,\text{mm}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(2.40\times10^6)}{(3.20-7.20)\times10^6} = 1.200$$

Then $2\theta_m = 50.194° \quad \text{and} \quad 230.194°$

or $\theta_m = 25.1° \text{ and } 115.1°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\max,\min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + I_{xy}^2}$$

Then

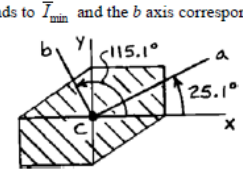
$$\bar{I}_{\max,\min} = \left[\frac{3.20+7.20}{2} \pm \sqrt{\left(\frac{3.20-7.20}{2}\right)^2 + (2.40)^2}\right]\times10^6\,\text{mm}^4$$

$$= (5.20 \pm 3.1241)\times10^6\,\text{mm}^4$$

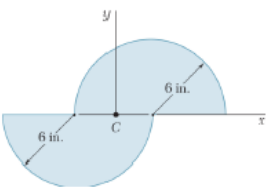
or $\bar{I}_{\max} = 8.32\times10^6\,\text{mm}^4$ ◀

and $\bar{I}_{\min} = 2.08\times10^6\,\text{mm}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\min}$ and the b axis corresponds to $\bar{I}_{\max}$.



9.87-11y12va



PROBLEM 9.87

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.73.

SOLUTION

From Problem 9.81: $\bar{I}_x = 324\pi \text{ in}^4 \qquad \bar{I}_y = 648\pi \text{ in}^4$

Problem 9.73: $\bar{I}_{xy} = 864 \text{ in}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(864)}{324\pi - 648\pi}$$

$$= 1.69765$$

Then $2\theta_m = 59.500° \quad \text{and} \quad 239.500°$

or $\theta_m = 29.7° \quad \text{and} \quad 119.7°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\max,\min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

Then

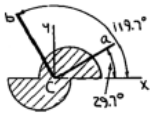
$$\bar{I}_{\max,\min} = \frac{324\pi + 648\pi}{2} \pm \sqrt{\left(\frac{324\pi - 648\pi}{2}\right)^2 + 864^2}$$

$$= (1526.81 \pm 1002.75) \text{ in}^4$$

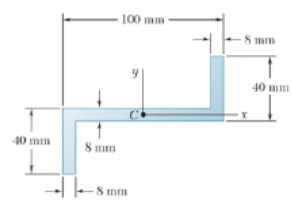
or $\bar{I}_{\max} = 2530 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 524 \text{ in}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\min}$ and the b axis corresponds to $\bar{I}_{\max}$.



9.88-11y12va



PROBLEM 9.88

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.75.

SOLUTION

From Problem 9.82: $\bar{I}_x = 252{,}757 \text{ mm}^4$

$\bar{I}_y = 1{,}752{,}789 \text{ mm}^4$

Problem 9.75: $\bar{I}_{xy} = 471{,}040 \text{ mm}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y}$$

$$= -\frac{2(471{,}040)}{252{,}757 - 1{,}752{,}789}$$

$$= 0.62804$$

Then $2\theta_m = 32.130° \quad \text{and} \quad 212.130°$

or $\theta_m = 16.07° \text{ and } 106.1°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\max,\min} = \frac{\bar{I}_x - \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

Then

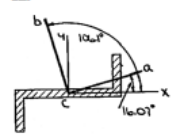
$$\bar{I}_{\max,\min} = \frac{252{,}757 + 1{,}752{,}789}{2} \pm \sqrt{\left(\frac{252{,}757 - 1{,}752{,}789}{2}\right)^2 + 471040^2}$$

$$= (1{,}002{,}773 \pm 885{,}665) \text{ mm}^4$$

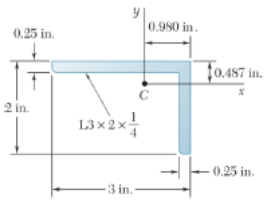
or $\bar{I}_{\max} = 1.888 \times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\min} = 0.1171 \times 10^6 \text{ mm}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\min}$ and the b axis corresponds to $\bar{I}_{\max}$.



9.89-11y12va



PROBLEM 9.89

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The $L3 \times 2 \times \frac{1}{4}$-in. angle cross section of Problem 9.74.

SOLUTION

From Problem 9.83: $\bar{I}_x = 0.390 \text{ in}^4 \qquad \bar{I}_y = 1.09 \text{ in}^4$

Problem 9.74: $\bar{I}_{xy} = -0.37983 \text{ in}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(-0.37983)}{0.390 - 1.09} = -1.08523$$

Then $2\theta_m = -47.341° \text{ and } 132.659°$

or $\theta_m = -23.7° \text{ and } 66.3°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\max,\min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

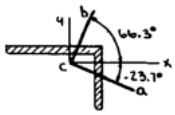
Then

$$\bar{I}_{\max,\min} = \frac{0.390 + 1.09}{2} \pm \sqrt{\left(\frac{0.390 - 1.09}{2}\right)^2 + (-0.37983)^2} = (0.740 \pm 0.51650)^2 \text{ in}^4$$

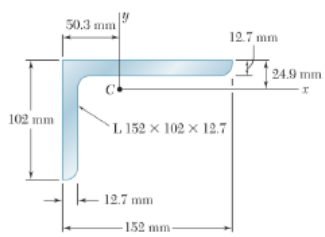
or $\bar{I}_{\max} = 1.257 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 0.224 \text{ in}^4$ ◀

By inspection, the a axis corresponds to $\bar{I}_{\min}$ and the b axis corresponds to $\bar{I}_{\max}$.



9.90-11y12va



PROBLEM 9.90

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The $L152 \times 102 \times 12.7$-mm angle cross section of Problem 9.78.

SOLUTION

From Figure 9.13B: $\bar{I}_x = 2.59 \times 10^6 \text{ mm}^4$

$\bar{I}_y = 7.20 \times 10^6 \text{ mm}^4$

Problem 9.78: $\bar{I}_{xy} = 2.54 \times 10^6 \text{ mm}^4$

Now Eq. (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(2.54 \times 10^6)}{(2.59 - 7.20) \times 10^6} = 1.10195$$

Then $2\theta_m = 47.777° \text{ and } 227.777°$

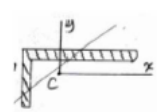
or $\theta_m = 23.9° \text{ and } 113.9°$ ◀

Also Eq. (9.27):

$$\bar{I}_{\max,\min} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

Then

$$\bar{I}_{\max,\min} = \left[\frac{2.59 + 7.20}{2} \pm \sqrt{\left(\frac{2.59 - 7.20}{2}\right)^2 + 2.54^2}\right] \times 10^6 \text{ mm}^4 = (4.895 \pm 3.4300) \times 10^6 \text{ mm}^4$$

or $\bar{I}_{\max} = 8.33 \times 10^6 \text{ mm}^4$ ◀

and $\bar{I}_{\min} = 1.465 \times 10^6 \text{ mm}^4$ ◀